# BIRTHDAY PARADOX

SPRING SEMESTER 2024

https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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## 1 Introduction

In probability theory, the birthday paradox concerns the probability, $p(n)$, that, in a group of $n$ randomly chosen people, at least one pair has the same birthday. The paradox, which is described below in more details, is a good illustrations how intuition can lead one astray in probability theory.

For simplicity we assume that there are $M = 365$ possible birthdays in a year that are equally likely.

The probability $p(n)$ that a pair of people in a group has the same birthday is 0 when there is one person in the group. The probability $p(n)$ is 1 when the number of people exceeds 365 (since there are only 365 possible birthdays). It seems plausible that the probability that a pair of people in a group has the same birthday is $\approx \frac{1}{2}$ when the number of people is close to $\frac{M}{2} \approx 182$.

The problem is to compute the probability, $p(n)$ that in a group of $n$ people, at least two have the same birthday. However, it is simpler to calculate $p_1(n)$, the probability that no two people in the group have the same birthday. Since the two events are mutually exclusive,

$$p(n) = 1 - p_1(n).$$

Let's consider the probability of an elementary event, $P(i)$, that the person No. $i$ in the group is not sharing his/her birthday with any of the previous $i-1$ people. For Person 1, there are no previously analyzed people. Therefore, the probability, $P(1)$ is 1. The probability of 1 can also be written as

$$P(1) = 365/365 = 1 - 0/365,$$

for reasons that will become clear below.

For Person 2, the only previously analyzed people is Person 1. The probability, $P(2)$, that Person 2 has a different birthday than Person 1 is

$$P(2) = 364/365 = 1 - 1/365.$$

Indeed, if Person 2 was born on any of the other 364 days of the year, Persons 1 and 2 do not share the same birthday.

Similarly, if Person 3 is born on any of the 363 days of the year other than the birthdays of Persons 1 and 2, Person 3 will not share their birthday. This makes the probability $P(3)$

$$P(3) = 363/365 = 1 - 2/365.$$

This analysis continues until the last person in the group, Person $n$ is reached, whose probability of not sharing his/her birthday with $(n-1)$ people analyzed before, $P(n)$, is

$$P(n) = \frac{365 - n + 1}{365} = 1 - \frac{n-1}{365}.$$

The probability that no two people in the group have the same birthday, $p_1(n)$, is equal to the product of individual probabilities:

$$p_1(n) = P(1) \cdot P(2) \cdot P(3) \dots P(n-1) \cdot P(n)$$

or

$$p_1(n) = 1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \cdots \times \left(1 - \frac{n-1}{365}\right) = \prod_{k=0}^{n-1} \left(1 - \frac{k}{365}\right).$$

Alternatively,

$$p_1(n) = \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n} = \frac{365!}{365^n (365-n)!}.$$

Returning to a general case of $M$ days in a year, $p(n, M)$ that in a group of $n$ people, at least two have the same birthday is as follows:

$$p(n, M) = 1 - \frac{M!}{M^n (M-n)!}.$$

Later in the class we obtain approximate analytic expressions for $p(n, M)$ in much more "usable form". For reference, for the case when $n \ll M$, we'll get:

$$p(n, M) \approx 1 - \exp(-(n-1)^2/(2M)).$$

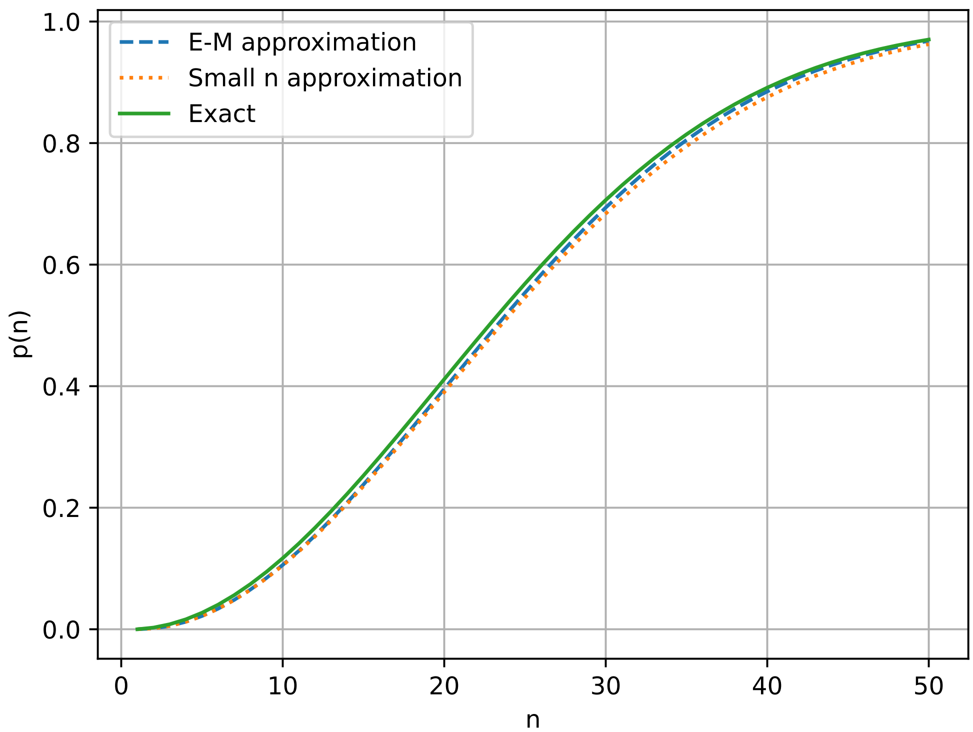


Figure 1: Top: the probability, $p(n)$, that in a group of $n$ people, at least two have the same birthday. Bottom: relative errors of two approximations.