PERTURBATION METHODS

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https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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1 Introduction

Perturbation theory is a collection of methods for obtaining approximate solutions to problems involving a small parameter ε. These methods are very powerful, thus sometimes it is actually advisable to introduce a parameter ε temporarily into a difficult problem having no small parameter, and then finally to set $\varepsilon = 1$ to recover the original problem. The approach of perturbation theory is to decompose a tough problem into an (infinite) number of relatively easy ones. The perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections.

We classify perturbation solutions into two types. A basic feature of *regular perturbation* problems is that the exact solution for small but nonzero ε smoothly approaches the unperturbed solution as $\varepsilon \to 0$.

We define a *singular perturbation* problem as one whose solution for $\varepsilon = 0$ is fundamentally different in character from the "neighboring" solutions obtained in the limit $\varepsilon \to 0$.

2 Regular perturbation theory

2.1 An example of perturbative analysis: roots of a polynomial

We consider first an elementary example to introduce the ideas of regular perturbation theory. Let us find approximations to the roots of the following equation.

$$x^5 - 16x + 1 = 0. \tag{1}$$

For the reference, Eq. (1) has three real roots and two complex conjugate ones. The numerical values of the roots[1] are $x_1 = -2.01533$, $x_2 = 1.98406$, $x_3 = 0.0625001$, and $x_{4,5} = -0.0156155 \pm 2.0003i$.

As it stands, this problem is not a perturbation problem because there is no small parameter in Eq. (1). In general it may not be easy to convert a particular problem into a tractable perturbation problem, but in the present case the trick is to replace the constant term in Eq. (1) with a parameter:

$$x^5 - 16x + \varepsilon = 0. \tag{2}$$

When $\varepsilon = 1$, the original Eq. (1) is reproduced.

We consider the values of roots to be functions of ε. We further assume a perturbation series in powers of ε:

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n. \tag{3}$$

To obtain the first term in this series, we set $\varepsilon = 0$ in Eq. (2) and factor it as following

$$x^5 - 16x = 0 \;\rightarrow\; x(x^2-4)(x^2+4) = 0 \;\rightarrow\; x(x-2)(x+2)(x-2i)(x+2i) = 0. \tag{4}$$

Thus, in the zeroth-order perturbation theory the equation's roots are:

$$x_m = \pm 2,\, 0,\, \pm 2i, \quad m = 1, \ldots, 5. \tag{5}$$

A second-order perturbation approximation to the first of these roots consists of writing

$$x_1 = -2 + a_1 \varepsilon + a_2 \varepsilon^2, \tag{6}$$

substituting this expression into Eq. (2), and neglecting powers of ε beyond ε^2. [2] The result is

$$(1 + 64a_1)\,\varepsilon + (-80a_1 + 64a_2)\,\varepsilon^2 = 0. \tag{7}$$

[1] The roots can be determined using a computer algebra system, e.g. with the following *Mathematica* command: `NSolve[x^5-16x+1==0,x]`.

[2] Computer algebra systems are perfectly suited for the tasks like this. E.g. the command `ord=3; li = CoefficientList[Series[x^5-16*x+ε/.x -> Sum[a[n]*ε^n, {n,0,ord}], {ε,0,ord}], ε]; sol = Solve[li==0]; r = Sum[a[n]ε^n, {n,0,ord}]/.sol; r/.ε->1` does the job.

Since ε is a variable, we conclude that the coefficient of each power of ε in Eq. (7) are separately equal to zero.

This gives a sequence of equations for the expansion coefficients a_i:

$$1 + 64a_1 = 0, \quad -80a_1^2 + 64a_2 = 0, \tag{8}$$

with the solutions

$$a_1 = -\frac{1}{64}, \quad a_2 = \frac{5}{4}a_1^2 = \frac{5}{16384}. \tag{9}$$

Therefore, the perturbation expansion for the root x_1 is

$$x_1 = -2 - \frac{1}{64}\varepsilon + \frac{5}{16384}\varepsilon^2. \tag{10}$$

If we now set $\varepsilon = 1$, we obtain $x_1 = -2.01532$ accurate to better than 10^{-5}. The same procedure gives

$$x_2 = 2 - \frac{1}{64}\varepsilon - \frac{5}{16384}\varepsilon^2 = 1.98407 \tag{11}$$

$$x_3 = \frac{1}{16}\varepsilon = 0.06250 \tag{12}$$

$$x_4 = -2i - \frac{1}{64}\varepsilon - \frac{5i}{16384}\varepsilon^2 = -.015625 - 2.00031i \tag{13}$$

$$x_5 = 2i - \frac{1}{64}\varepsilon + \frac{5i}{16384}\varepsilon^2 = -.015625 + 2.00031i \tag{14}$$

This example illustrates the three steps of perturbative analysis:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

Step 1 is sometimes ambiguous because there may be many ways to introduce a small parameter in the equation. It is preferable to introduce ε in such a way that the zeroth-order solution (the leading term in the perturbation series) is obtainable as a closed-form analytic expression. Step 1 may be omitted when the original problem already has a small parameter if a perturbation series can be developed in powers of that parameter.

2.2 Perturbative solution of a boundary-value problem

Let's apply the perturbation theory to a boundary value problem for an ordinary differential equation.

Consider the following nonlinear two-point boundary-value problem:

$$y'' + y = \frac{\cos(x)}{2+2y^2}, \qquad y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 1. \tag{15}$$

As a first step, we convert Eq. (15) into a perturbation problem by introducing ε in the right side of the equation. Then we obtain a first-order approximation to the answer. Finally, we return to the original equation by assigning $\varepsilon = 1$.

$$y'' + y = \varepsilon \frac{\cos(x)}{2+2y^2}. \tag{16}$$

We assume a perturbation expansion for $y(x)$ in the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \tag{17}$$

where $y_0(0) = 2$, $y_0\left(\frac{\pi}{2}\right) = 1$, $y_n(0) = y_n\left(\frac{\pi}{2}\right) = 0$ for $n \geq 1$.

The zeroth-order problem $y'' + y = 0$ is obtained by setting $\varepsilon = 0$. The solution that satisfies the boundary conditions is

$$y_0(x) = 2\cos(x) + \sin(x). \tag{18}$$

The first order problem is obtained by substituting Eqs. (17), (18) into Eq. (16) and equating the coefficient of ε in the left and right hand sides of the equation:

$$y_1'' + y_1 = \frac{\cos(x)}{2+2y_0^2} = \frac{\cos(x)}{7+3\cos(2x)+4\sin(2x)}. \tag{19}$$

Eq. (19) is linear inhomogeneous differential equation. It can be solved by variation of parameters. The solution of Eq. (19) is as following:

$$y_1(x) = a(x)\cos(x) + b(x)\sin(x), \tag{20}$$

where $a(0) = 0$ and $b\left(\frac{\pi}{2}\right) = 0$,

$$\begin{aligned} a(x) &= -\int_0^x \frac{\cos(y)\sin(y)}{7 + 3\cos(2y) + 4\sin(2y)}\,\mathrm{d}y \\ &= \frac{1}{300}\Bigg(-24x - 14\sqrt{6}\arctan\left(\sqrt{\frac{2}{3}}\right) + 14\sqrt{6}\arctan\left(\sqrt{\frac{2}{3}}(1 + \tan(x))\right) \\ &\qquad - 9\ln(10) + 9\ln\Big(7 + 3\cos(2x) + 4\sin(2x)\Big)\Bigg), \end{aligned} \tag{21}$$

$$\begin{aligned} b(x) &= \int_{\frac{\pi}{2}}^x \frac{\cos^2(y)}{7 + 3\cos(2y) + 4\sin(2y)}\,\mathrm{d}y \\ &= \frac{1}{300}\Bigg(18x - 9\pi - \sqrt{6}\pi + 2\sqrt{6}\arctan\left(\sqrt{\frac{2}{3}}(1 + \tan(x))\right) \\ &\qquad - 24\ln(2) + 12\ln\Big(7 + 3\cos(2x) + 4\sin(2x)\Big)\Bigg). \end{aligned} \tag{22}$$

The graph of the first order solution

$$y(x) = y_0(x) + y_1(x) \tag{23}$$

is presented in Fig. 1 together with the numerical solution of Eq. (15).

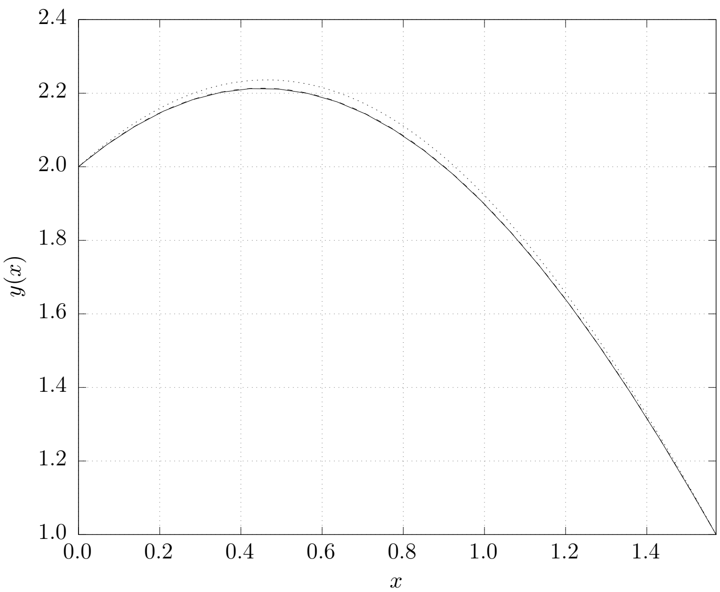

Figure 1: The graphs of the numerical solution of the boundary value problem Eq. (15) (solid line), perturbation solution Eq. (23) (dashed line), and perturbation solution Eq. (18) (dotted line). Perturbation approximation Eq. (23) is practically indistinguishable on the graph from the numerical solution.

3 Singular perturbation theory

3.1 The method of dominant balance

Let's introduce an approach for attacking "unsolvable" problems that is called the *method of dominant balance*.

Suppose you are given an equation in the form

$$A + B + C = 0, \tag{24}$$

where A, B, C are different terms in the equation. These terms could represent elements of a polynomial equation (e.g. x^5) or could represent terms in an ordinary differential equation or terms in a nonlinear partial differential equation. It is almost invariably the case that two of the terms are larger than the third one. For example, let's assume that A, C are larger than B. In this case we can then "approximate" the original equation by neglecting B entirely and consider that A is *balancing* C, that is solving the equation

$$A + C = 0. \tag{25}$$

We can then check that this reduced equation is "consistent", by taking the solution to the reduced equation, plugging it back into the original equation, and verifying that the neglected terms are indeed smaller than the terms we have kept. If they are, our solution is "consistent". If not the solution is "inconsistent" and we must consider another *dominant balance*.

3.2 Roots of a polynomial, II

To illustrate the work of the method of dominant balance, let's return to Eq. (1) and convert it to a perturbation problem as following:

$$\varepsilon x^5 - 16x + 1 = 0. \tag{26}$$

We begin by setting $\varepsilon = 0$ to obtain the unperturbed problem $-16x + 1 = 0$, with the solution $x = \frac{1}{16}$. Note that the unperturbed equation has only one root while the original equation has five roots. Part of the exact solution ceases to exist when $\varepsilon = 0$. This abrupt change in the character of the solution, namely the disappearance of four roots when $\varepsilon = 0$, implies that Eq. (26) is a *singular perturbation problem*.

The explanation for this behavior is that the four missing roots tend to ∞ as $\varepsilon \to 0$. Thus, for those roots it is no longer valid to neglect εx^5 compared with $-16x + 1$ in the limit $\varepsilon \to 0$.

To track down the four missing roots we first estimate their orders of magnitude as $\varepsilon \to 0$. We do this by considering all possible dominant balances between pairs of terms in Eq. (26). There are three terms in so there are three pairs to consider:

1. Suppose $16x \sim 1$ is the dominant balance. This is a consistent assumption because the other term in the equation, $\sim \varepsilon$, is negligible compared with x and 1, and we recover the root of the unperturbed equation $16x - 1 = 0$. We may assume a regular perturbation expansion Eq. (3) for this root, as we did in Sec. 2.1, and obtain:

$$x = \frac{1}{16} + \frac{1}{16777216}\varepsilon + \frac{5}{17592186044416}\varepsilon^2 + \frac{35}{18446744073709551616}\varepsilon^3 + \dots \tag{27}$$

2. Suppose $\varepsilon x^5 \sim -1$ is the dominant balance. Then $x \sim \varepsilon^{-\frac{1}{5}}$.

   For this balance to be consistent the neglected term, x, must be smaller than the terms we have kept. But it is not! In the limit $\varepsilon \to 0$, the size of the neglected term, $\sim \varepsilon^{-\frac{1}{5}} \to \infty$, whereas the two terms we have kept are of order unity. Therefore this is not a consistent balance.

3. Suppose $\varepsilon x^5 \sim 16x$ is the dominant balance. Then $x \sim \varepsilon^{-\frac{1}{4}}$.

   The size of the neglected term, 1, is much smaller than the size of the terms that we have kept $\sim \varepsilon^{-\frac{1}{4}}$ as $\varepsilon \to 0$, so this balance is self consistent.

Thus, the magnitudes of the four missing roots are $\sim \varepsilon^{-\frac{1}{4}}$ as $\varepsilon \to 0$. This result suggests a scale transformation for the variable x:

$$x = \varepsilon^{-\frac{1}{4}} y. \tag{28}$$

The substituting of Eq. (28) into Eq. (26) gives

$$y^5 - 16y + \varepsilon^{\frac{1}{4}} = 0. \tag{29}$$

This is now a regular perturbation problem for y in the parameter $\varepsilon^{\frac{1}{4}}$ because the unperturbed problem $y^5 - 16y = 0$ has five roots. No roots disappear in the limit $\varepsilon \to 0$.

The perturbative corrections to these roots may be found by assuming a regular perturbation expansion in powers of $\varepsilon^{\frac{1}{4}}$.

$$y(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^{\frac{n}{4}}. \tag{30}$$

It would not be possible to match powers in an expansion having only integral powers of ε.

The results are:

$$y_1 = -2 - \frac{1}{64}\varepsilon^{\frac{1}{4}} + \frac{5}{16384}\varepsilon^{\frac{1}{2}} - \frac{5}{524288}\varepsilon^{\frac{3}{4}} + \ldots, \tag{31}$$

$$y_2 = 2 - \frac{1}{64}\varepsilon^{\frac{1}{4}} - \frac{5}{16384}\varepsilon^{\frac{1}{2}} - \frac{5}{524288}\varepsilon^{\frac{3}{4}} - \ldots, \tag{32}$$

$$y_3 = \frac{1}{16}\varepsilon^{\frac{1}{4}} + \frac{1}{16777216}\varepsilon^{\frac{5}{4}} + \frac{5}{17592186044416}\varepsilon^{\frac{9}{4}} + \ldots, \tag{33}$$

$$y_4 = -2i - \frac{1}{64}\varepsilon^{\frac{1}{4}} - \frac{5i}{16384}\varepsilon^{\frac{1}{2}} + \frac{5}{524288}\varepsilon^{\frac{3}{4}} + \ldots, \tag{34}$$

$$y_5 = 2i - \frac{1}{64}\varepsilon^{\frac{1}{4}} + \frac{5i}{16384}\varepsilon^{\frac{1}{2}} + \frac{5}{524288}\varepsilon^{\frac{3}{4}} - \ldots. \tag{35}$$

Note that only the second, fifths, ninths, ... coefficient in Eq. (33) for y_3, is non-vanishing. Thus we have also reproduced the regular perturbation series of Eq. (27).

3.3 Boundary layers

We now discuss the perturbative method that is called boundary-layer theory. It is used for solving a differential equation whose highest derivative is multiplied by the perturbing parameter ε. A boundary layer is a narrow region where the solution of a differential equation changes rapidly. By definition, the thickness of a boundary layer must approach 0 as $\varepsilon \to 0$.

Example 1. Exactly soluble boundary-layer problem Consider the following boundary-value problem whose solution, as we see shortly, exhibits boundary-layer structure.

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \qquad y(0) = 0,\ y(1) = 1. \tag{36}$$

Eq. (36) is a differential equation with constant coefficients. Looking for the solution in the form

$$y(x) \sim e^{rx}, \tag{37}$$

we obtain the following quadratic equation for r:

$$r^2 + \left(1 + \frac{1}{\varepsilon}\right) r + \frac{1}{\varepsilon} = 0. \tag{38}$$

$$r_{1,2} = -\frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right) \pm \sqrt{\frac{1}{4}\left(1 + \frac{1}{\varepsilon}\right)^2 - \frac{1}{\varepsilon}} = -\frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right) \pm \frac{1}{2}\left(1 - \frac{1}{\varepsilon}\right). \tag{39}$$

$$r_1 = -1, \quad r_2 = -\frac{1}{\varepsilon}. \tag{40}$$

The general solution of Eq. (36) is

$$y(x) = C_1 e^{-x} + C_2 e^{-\frac{x}{\varepsilon}}, \tag{41}$$

where C_1 and C_2 are integration constants.

The boundary condition at $x = 0$, $y(0) = 0$, gives

$$C_1 = -C_2. \tag{42}$$

The boundary condition at $x = 1$, $y(1) = 0$, gives

$$C_1 = \left(e^{-1} - e^{-\frac{1}{\varepsilon}}\right)^{-1}. \tag{43}$$

Thus, the exact solution of the boundary value problem Eq. (36) is:

$$y(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}}. \tag{44}$$

The graph of the solution Eq. (44) is presented in Fig. 2. For small ε the solution Eq. (44) is slowly varying for $\varepsilon \ll x \le 1$. However, on the small interval $0 < x \le O(\varepsilon)$ it undergoes a rapid change.

The small interval of rapid change is called a *boundary layer*. The region of slow variation of $y(x)$ is called the *outer region*. The boundary-layer region is also called the *inner region*.

Boundary-layer theory is a collection of methods for solving differential equations whose solutions exhibit boundary-layer structure.

There are two standard approximations that one makes in boundary-layer theory:

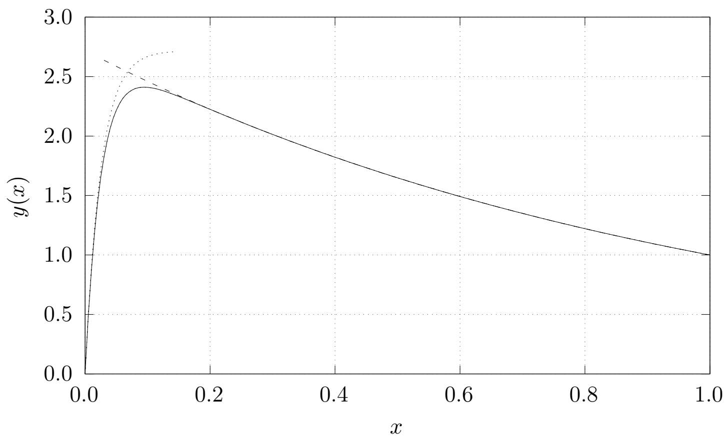

Figure 2: The graphs of the solution of the boundary value problem Eq. (36) for $\varepsilon = 0.025$ (solid line). The inner and outer solutions, Eq. (56) and Eq. (47) respectively, are shown in dashed and dotted lines. (The uniform approximation to the exact solution, $y_u(x)$, Eq. (58), is also plotted but is indistinguishable from the exact solution.)

1. In the outer region (away from a boundary layer) $y(x)$ is slowly varying, so it is valid to neglect any derivatives of $y(x)$ which are multiplied by ε.
2. Inside a boundary layer the derivatives of $y(x)$ are large, but the boundary layer is so narrow that we may approximate the coefficient functions of the differential equation by constants.

In every region the solution of the approximate equation will contain one or more unknown constants of integration. These constants are then determined from the boundary or initial conditions using the technique of asymptotic matching.

Example 2. Using boundary-layer theory for the same exactly soluble problem Let's now use the boundary-layer theory to solve the problem Eq. (36). In the outer region ($\varepsilon = 0$) we have the following first order differential equation:

$$y' + y = 0 \tag{45}$$

with the general solution

$$y_{\text{out}}(x) = Ce^{-x}, \tag{46}$$

where C is the integration constant. Since no choice of C can satisfy the boundary condition $y(0) = 0$, we conclude that the outer region is adjacent to $x = 1$. We determine the integration constant from the boundary condition $y(1) = 1$:

$$y(1) = Ce^{-1} = 1 \quad \rightarrow \quad C = e,\ y_{\text{out}}(x) = e^{1-x}. \tag{47}$$

In the inner region, which we now know is adjacent to $x = 0$, $y'', y' \gg y$, thus the differential equation in the inner region is as following:

$$\varepsilon y'' + y' = 0. \tag{48}$$

Eq. (48) can be solved by introducing a temporary unknown function $v(x) = y'(x)$. We have the following differential equation for v

$$v' + \frac{1}{\varepsilon}v = 0, \tag{49}$$

with the solution

$$v(x) = y'(x) = C_0 e^{-\frac{x}{\varepsilon}}. \tag{50}$$

Integrating once again, we obtain the inner solution as following

$$y_{\text{in}}(x) = C_1 e^{-\frac{x}{\varepsilon}} + C_2, \tag{51}$$

where C_1 and C_2 are two integration constants. We determine those from the boundary condition at $x = 0$ and from matching the inner and the outer solutions.

The boundary condition $y_{\text{in}}(0) = 0$ gives

$$C_1 + C_2 = 0 \quad \rightarrow \quad C_1 = -C_2, \tag{52}$$

that is

$$y_{\text{in}}(x) = C_1 e^{-\frac{x}{\varepsilon}} - C_1, \tag{53}$$

To determine the remaining integration constant, C_1 we require that the limit of the outer solution, $y_{\text{out}}(x)$, as $x \rightarrow 0$ be the same as the limit of the inner solution, $y_{\text{in}}(x)$, as $x \rightarrow \infty$:

$$y_{\text{lim}} = \lim_{x\rightarrow 0} y_{\text{out}}(x) = e, \quad y_{\text{lim}} = \lim_{x\rightarrow\infty} y_{\text{in}}(x) = -C_1, \tag{54}$$

thus

$$C_1 = -e \tag{55}$$

and

$$y_{\text{in}}(x) = -e^{1-\frac{x}{\varepsilon}} + e. \tag{56}$$

Finally we need to construct the so called *uniform approximation* for the solution for $0 \leq x \leq 1$:

$$y_{\text{u}} = y_{\text{in}}(x) + y_{\text{out}}(x) - y_{\text{lim}}, \tag{57}$$

where $y_{\text{lim}}(x)$ is the common limit of $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ given by Eq. (54).

$$y_{\text{u}} = e^{1-x} - e^{1-\frac{x}{\varepsilon}} = e\left(e^{-x} - e^{-\frac{x}{\varepsilon}}\right) \tag{58}$$

The uniform approximation Eq. (58) is compared to the exact solution and to the inner and the outer solutions in Fig. 2.

Example 3. Using dominant balance approach for the exactly soluble problem Let's re-scale x,

$$x = \varepsilon^{\alpha}\,\xi, \quad \frac{\mathrm{d}}{\mathrm{d}x} = \varepsilon^{-\alpha}\,\frac{\mathrm{d}}{\mathrm{d}\xi}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \varepsilon^{-2\alpha}\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \tag{59}$$

where α is a parameter that we determine later.

$$\varepsilon^{1-2\alpha}\,\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \varepsilon^{-\alpha}\,\frac{\mathrm{d}y}{\mathrm{d}\xi} + \varepsilon^{1-\alpha}\,\frac{\mathrm{d}y}{\mathrm{d}\xi} + y = 0. \tag{60}$$

There are four terms in Eq. (60), therefore there are six pairs to consider for dominant balance.

- Suppose that the first and the second terms in Eq. (60) are the dominant balance:

$$\varepsilon^{1-2\alpha}\,\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} \sim \varepsilon^{-\alpha}\,\frac{\mathrm{d}y}{\mathrm{d}\xi}. \tag{61}$$

  This is possible if $1-2\alpha = -\alpha$, i.e. $\alpha = 1$. The balance is consistent since the other two terms in the equation, which now takes the form

$$\frac{1}{\varepsilon}\,\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \frac{1}{\varepsilon}\,\frac{\mathrm{d}y}{\mathrm{d}\xi} + \frac{\mathrm{d}y}{\mathrm{d}\xi} + y = 0, \tag{62}$$

  are much smaller that the terms we keep. The equation in the dominant balance approximation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \frac{\mathrm{d}y}{\mathrm{d}\xi} = 0, \tag{63}$$

  with the solution

$$y(\xi) = C_1 e^{-\xi} + C_2. \tag{64}$$

  As a function of the "original" x,

$$y(x) = C_1 e^{-\frac{x}{\varepsilon}} + C_2, \tag{65}$$

  which is exactly the solution Eq. (51).

- Suppose that the first and the second terms in Eq. (60) are the dominant balance:

$$\varepsilon^{-\alpha}\,\frac{\mathrm{d}y}{\mathrm{d}\xi} \sim y = 0. \tag{66}$$

This is possible if $\alpha = 0$. The balance is consistent since the other two terms in the equation, which now takes the form

$$\varepsilon \frac{d^2 y}{d\xi^2} + \frac{dy}{d\xi} + \varepsilon \frac{dy}{d\xi} + y = 0, \tag{67}$$

are much smaller that the terms we keep. The equation in the dominant balance approximation is

$$\frac{dy}{d\xi} + y = 0, \tag{68}$$

with the solution

$$y(x) = Ce^{-x}, \tag{69}$$

which is exactly the solution Eq. (46).

- All other pairs of terms in Eq. (60) produce inconsistent balances.

Repeating the steps of matching solutions Eq. (69) and Eq. (65) with each other and with the boundary conditions, we obtain the same uniform solution Eq. (58).

Example 4. Using boundary-layer theory for a nonlinear equation Consider now the following nonlinear boundary value problem:

$$\varepsilon y'' + y' + e^{-y} = 0, \qquad y(0) = 2,\ y(1) = 0. \tag{70}$$

In the outer region ($\varepsilon = 0$) we have the following first order differential equation:

$$y' + e^{-y} = 0 \tag{71}$$

with the general solution

$$y_{\text{out}}(x) = \ln(C - x), \tag{72}$$

where C is the integration constant. For now let's suppose that the outer region is adjacent to $x = 1$. We determine the integration constant from the boundary condition $y(1) = 0$:

$$y(1) = \ln(C - 1) = 0 \quad \rightarrow \quad C = 2,\ y_{\text{out}}(x) = \ln(2 - x). \tag{73}$$

In the inner region, which we now know is adjacent to $x = 0$, $y'', y' \gg y$, therefore the differential equation in the inner region is as following:

$$\varepsilon y'' + y' = 0. \tag{74}$$

Eq. (74) is the same as Eq. (48) that we already solved. Thus the inner solution as follows:

$$y_{\text{in}}(x) = C_1 e^{-\frac{x}{\varepsilon}} + C_2, \tag{75}$$

where C_1 and C_2 are two integration constants. We determine those from the boundary condition at $x = 0$ and from matching the inner and the outer solutions.

The boundary condition $y_{\text{in}}(0) = 2$ gives

$$C_1 + C_2 = 2 \quad \rightarrow \quad C_2 = 2 - C_1, \tag{76}$$

that is

$$y_{\text{in}}(x) = C_1 e^{-\frac{x}{\varepsilon}} - C_1 + 2. \tag{77}$$

To determine the remaining integration constant, C_1 we require that the limit of the outer solution, $y_{\text{out}}(x)$, as $x \rightarrow 0$ be the same as the limit of the inner solution, $y_{\text{in}}(x)$, as $x \rightarrow \infty$:

$$y_{\text{lim}} = \lim_{x\rightarrow 0} y_{\text{out}}(x) = \ln(2), \quad y_{\text{lim}} = \lim_{x\rightarrow\infty} y_{\text{in}}(x) = 2 - C_1, \tag{78}$$

thus

$$C_1 = 2 - \ln(2) \tag{79}$$

and

$$y_{\text{in}}(x) = (2 - \ln(2))\, e^{-\frac{x}{\varepsilon}} + \ln(2). \tag{80}$$

Finally we need to construct the *uniform approximation* for the solution for $0 \le x \le 1$:

$$y_{\text{u}} = y_{\text{in}}(x) + y_{\text{out}}(x) - y_{\text{lim}}, \tag{81}$$

where $y_{\text{lim}}(x)$ is the common limit of $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ given by Eq. (78).

$$y_{\text{u}} = \ln(2 - x) - (2 - \ln(2))\, e^{-\frac{x}{\varepsilon}}. \tag{82}$$

The uniform approximation Eq. (82) is compared to the exact solution and to the inner and the outer solutions in Fig. 3.

4 Van der Pol oscillator for large nonlinearity

The second order non-linear autonomous differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad \varepsilon > 0 \tag{83}$$

is called van der Pol equation. The parameter ε is positive and indicates the nonlinearity of the system. The equation models a non-conservative system in which energy is added to and subtracted from the system, resulting in a periodic motion called a *limit cycle*. In this section we obtain an approximation for the limit cycle for large ε, $\varepsilon \gg 1$.

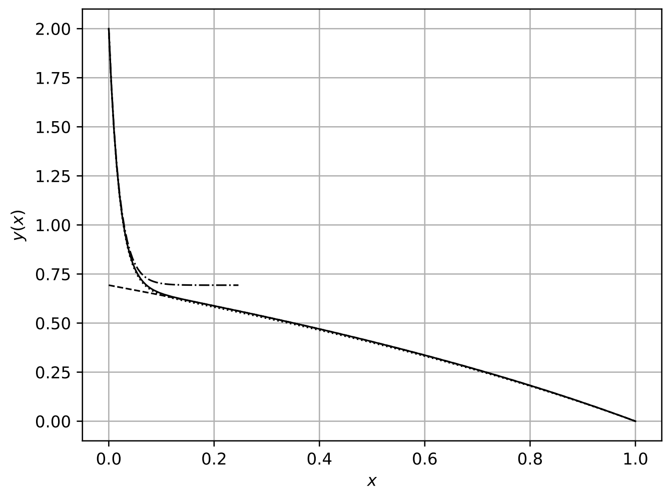

Figure 3: The graphs of the numerical solution of the boundary value problem Eq. (70) for $\varepsilon = 0.02$ (dotted line). The inner and outer solutions, Eq. (80) and Eq. (73) respectively, are shown as dot-dashed and dashed lines. The uniform approximation to the exact solution, $y_{\mathrm{u}}(x)$, Eq. (82), is plotted as solid line.

First, we define a small parameter, $\nu \equiv \varepsilon^{-1}$, $\nu \ll 1$. In terms of ν Eq. (83) takes the form:

$$\nu \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(x^2 - 1\right) \frac{\mathrm{d}x}{\mathrm{d}t} + \nu\, x = 0. \tag{84}$$

Next we re-scale t,

$$t = \nu^{\alpha}\, \tau, \quad \frac{\mathrm{d}}{\mathrm{d}t} = \nu^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau}, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \nu^{-2\alpha} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \tag{85}$$

where α is a parameter that we determine later.

$$\nu^{1-2\alpha} \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \nu^{-\alpha} \left(x^2 - 1\right) \frac{\mathrm{d}x}{\mathrm{d}\tau} + \nu\, x = 0. \tag{86}$$

Finally, we select α so that two of the three terms in Eq. (86) are of the same order in ν, and are larger than the remaining term. There are three pairs to consider:

1. Suppose that the first and the third terms in Eq. (86) balance:

   $$\nu^{1-2\alpha} \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} \sim -\nu\, x. \tag{87}$$

   i.e. that $1 - 2\alpha = 1$, or $\alpha = 0$.

   For this balance to be consistent the neglected term, $\left(x^2 - 1\right) \frac{\mathrm{d}x}{\mathrm{d}\tau}$, must be smaller than the terms we have kept. But it is not! In the limit $\nu \to 0$, the size of the neglected term is of order $\nu^0 \sim 1$, whereas the two terms we have kept are of order ν, $\nu \ll 1$. Therefore this is not a consistent balance.

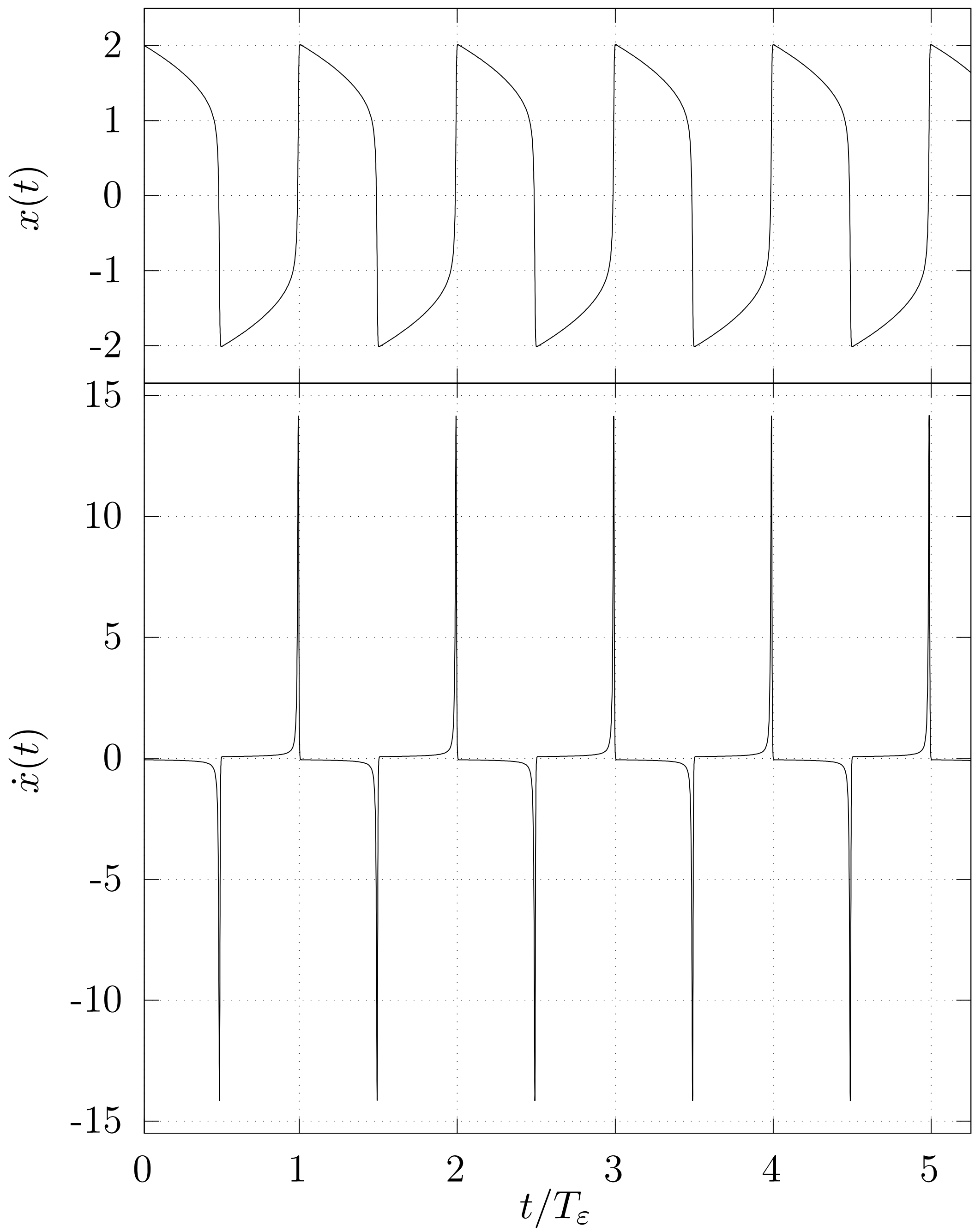

Figure 4: Numerical solution of van der Pol oscillator for $\varepsilon = 10$.

2. Suppose that the second and the third terms in Eq. (86) balance:

$$\nu^{-\alpha}\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau} \sim -\nu\, x. \tag{88}$$

i.e. that $\alpha = -1$.

The terms that we keep are both proportional to ν. The neglected term, $\nu^{1-2\alpha}\,\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}$ is proportional to ν^3, thus it is indeed much smaller that that the kept terms. Therefore this is a consistent balance.

Eq. (86) takes the form:

$$\nu^2\,\frac{\mathrm{d}^2x}{\mathrm{d}\tau_1^2} + \left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau_1} + x = 0, \tag{89}$$

where $\tau_1 \equiv \nu\, t = \frac{t}{\varepsilon}$ is called the *slow time*.

Neglecting the term that is proportional to ν^2 we get the following equation:

$$\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau_1} + x = 0, \tag{90}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}\tau_1} = \frac{x}{1-x^2}. \tag{91}$$

Equation (91) is a first order ordinary differential equation that can be integrated separating variables:

$$\left(\frac{1}{x}-x\right)\mathrm{d}x = \mathrm{d}\tau_1 \quad\longrightarrow\quad \ln|x| - \frac{x^2}{2} = \frac{t}{\varepsilon} + C_1, \tag{92}$$

where C_1 is an integration constant. This constant is irrelevant when we are interested how long it took to move from $x = x_i$ to $x = x_f$:

$$\Delta t = t(x_f) - t(x_i), \tag{93}$$

where

$$t(x) \stackrel{\text{def}}{=} \varepsilon\left(\ln|x| - \frac{x^2}{2}\right). \tag{94}$$

3. Suppose that the first and the second terms in Eq. (86) balance:

$$\nu^{1-2\alpha}\,\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} \sim -\nu^{-\alpha}\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau} \tag{95}$$

i.e. that $1-2\alpha=-\alpha$, or $\alpha=1$.

The terms that we keep are both proportional to ν^{-1}. The neglected term, νx is proportional to ν, thus it is indeed much smaller that that the kept terms. Therefore this is a consistent balance.

Eq. (86) takes the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau_2^2}+\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau_2}+\nu^2 x=0. \tag{96}$$

Here $\tau_2 \equiv \frac{t}{\alpha}=t\varepsilon$. Variable τ_2 is called the *fast time*.

Neglecting the term $\sim \nu^2$ we get the equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau_2^2}+\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}\tau_2}=0, \tag{97}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau_2}+\frac{x^3}{3}-x\right)=0. \tag{98}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau_2}+\frac{x^3}{3}-x=C_2, \tag{99}$$

where C_2 is an integration constant.

We now need to match the solutions Eq. (92) and (99) to find the integration constants C_1 and C_2. The motion proceeds according to Eq. (92) until it reaches $x=\pm 1$ where the speed $\frac{\mathrm{d}x}{\mathrm{d}t}$ is infinite. At this point the system undergoes a jump the which is described by the limit Eq. (99). We chose C_2 so that $x=1$ is an equilibrium point of Eq. (99):

$$\frac{\mathrm{d}x}{\mathrm{d}\tau_2}=0=C_2-\left(\frac{x^3}{3}-x\right)_{x=1}, \tag{100}$$

so that

$$C_2=-\frac{2}{3}. \tag{101}$$

Using the just determined value of C_2 we rewrite Eq. (99) as following[3]:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau_2}=-\frac{2}{3}-\frac{x^3}{3}+x=-\frac{1}{3}(x-1)^2(x+2). \tag{102}$$

[3]The factorization of the right hand side of Eq. (102) can be done using a computer algebra system, e.g. `Factor[-2/3 - x^3/3 + x]`

Therefore, the second equilibrium point, which corresponds to the landing point of the "jump" is at $x = -2$. That is, the jump goes from $x_i = 1$ to $x_f = -2$. Repeating the reasoning we find that the jump that starts at $x_i = -1$ ends at $x_f = 2$.

The time for the system to jump is negligible compared to the time spent in the "slow" motion in accordance with Eq. (93), (94). Thus, the period of van der Pol oscillator is:

$$T(\varepsilon) = 2\Big(t(1) - t(2)\Big) = \Big(3 - 2\ln 2\Big)\varepsilon. \tag{103}$$

References

[1] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.

[2] Michael P. Brenner. *Physical Mathematics*. http://isites.harvard.edu/fs/docs/icb.topic1311226.files/AM201_Brenner_CourseNotes_2010.pdf. [Online; accessed 2017-03-15]. 2010. URL: http://isites.harvard.edu/fs/docs/icb.topic1311226.files/AM201_Brenner_CourseNotes_2010.pdf.

[3] J. Kevorkian and J. D. Cole. *Perturbation Methods in Applied Mathematics*. Vol. 34. Applied Mathematical Sciences. Springer Verlag, 1981.