THE SURFACE AREA AND THE VOLUME OF N-DIMENSIONAL SPHERE

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1 Introduction

The volume of n-dimensional sphere of radius r is proportional to r^n,

$$V_n(r) = v(n)\, r^n, \tag{1}$$

where the proportionality constant, $v(n)$, is the volume of the n-dimensional unit sphere.

The surface area of n-dimensional sphere of radius r is proportional to r^{n-1}.

$$S_n(r) = s(n)\, r^{n-1}, \tag{2}$$

where the proportionality constant, $s(n)$, is the surface area of the n-dimensional unit sphere.

The n-dimensional sphere is a union of concentric spherical shells:

$$\mathrm{d}V_n(r) = S_n(r)\,\mathrm{d}r \tag{3}$$

Therefore the surface area and the volume of a sphere are related as following:

$$V_n(R) = \int_0^R S_n(r)\,\mathrm{d}r = s(n)\int_0^R r^{n-1}\,\mathrm{d}r = \frac{s(n)}{n} R^n. \tag{4}$$

The surface area and the volume of the unit sphere are related as following:

$$v(n) = \frac{s(n)}{n}. \tag{5}$$

Consider the integral

$$I_n = \int_{-\infty}^{\infty} e^{-x_1^2 - x_2^2 - \ldots - x_n^2}\,\mathrm{d}V_n = \int_0^{\infty} e^{-r^2}\,\mathrm{d}V_n(r), \tag{6}$$

where $\mathrm{d}V_n$ is the volume element in cartesian coordinates

$$\mathrm{d}V_n = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n \tag{7}$$

and

$$\mathrm{d}V_n(r) = s(n)\,r^{n-1}\,\mathrm{d}r \tag{8}$$

is the volume element in spherical coordinates.

Since the integrand in the first integral in Eq. (6) is a product of identical gaussians of one variable each,

$$I_n = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \right)^n = \left(\sqrt{\pi}\right)^n = \pi^{\frac{n}{2}}. \tag{9}$$

On the other hand, the second integral in Eq. (6), evaluated in spherical coordinates

$$I_n = \int_0^{\infty} e^{-r^2} s(n)\,r^{n-1}\,\mathrm{d}r = s(n) \int_0^{\infty} e^{-r^2} r^{n-1}\,\mathrm{d}r = \frac{s(n)}{2} \int_0^{\infty} e^{-r^2} \left(r^2\right)^{\frac{n}{2}-1}\,\mathrm{d}r^2 \tag{10}$$

$$= \frac{s(n)}{2} \int_0^{\infty} e^{-t}\,t^{\frac{n}{2}-1}\,\mathrm{d}t = \frac{s(n)}{2}\,\Gamma\left(\frac{n}{2}\right) \tag{11}$$

Comparing Eq. (9) and Eq. (11), we obtain:

$$s(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{12}$$

$$v(n) = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{13}$$

2 Coulomb's law in n-dimension

In three dimensions Coulomb's law takes the form

$$E_3(r) = \frac{1}{4\pi\epsilon_0}\,\frac{Q}{r^2}, \tag{14}$$

where E_3 is the magnitude of the electric field, ϵ_0 is the electric constant, and r is the distance from the point charge of charge Q. What is Coulomb's law look like in high dimensions?

We assume the Maxwell equations hold in any dimension. Hence Gauss law still holds:

$$\mathbf{\Phi} = \frac{Q}{\epsilon_0}, \tag{15}$$

where $\mathbf{\Phi}$ is the electric flux through a closed surface enclosing any volume, Q is the total charge enclosed within that volume.

Due to the spherical symmetry of the field created by a point charge,

$$\mathbf{\Phi} = E_n(r)\,S_n(r), \tag{16}$$

where $E_n(r)$ is the radial component of the electric field, $S_n(r)$ is the surface area of n-dimensional sphere of radius r.

$$S_n(r) = s(n)\,r^{n-1} = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\,r^{n-1}. \tag{17}$$

Comparing Eq. (15) and Eq. (16),

$$E_n(r) = \frac{Q}{\epsilon_0\,S_n(r)} = \frac{\Gamma\left(\frac{n}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\epsilon_0}\,\frac{Q}{r^{n-1}}. \tag{18}$$

References

[1] Wikipedia. *Volume of an n-ball — Wikipedia, The Free Encyclopedia*. Accessed: May 20, 2020. URL: https://en.wikipedia.org/wiki/Volume_of_an_n-ball.