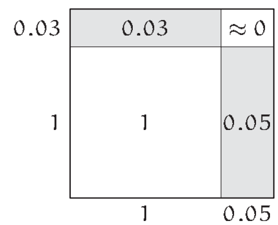
$$3.15 \times 7.21 = \underbrace{3 \times 7}_{\text{big part}} \times \underbrace{(1 + 0.05) \times (1 + 0.03)}_{\text{correction factor}}. \tag{5.4}$$

▶ *Can you find a picture for the correction factor?*



The correction factor is the area of a rectangle with width $1 + 0.05$ and height $1 + 0.03$. The rectangle contains one subrectangle for each term in the expansion of $(1 + 0.05) \times (1 + 0.03)$. Their combined area of roughly $1 + 0.05 + 0.03$ represents an 8% fractional increase over the big part. The big part is 21, and 8% of it is 1.68, so $3.15 \times 7.21 = 22.68$, which is within 0.14% of the exact product.

Problem 5.6 Picture for the fractional error
What is the pictorial explanation for the fractional error of roughly 0.15%?

Problem 5.7 Try it yourself
Estimate 245×42 by rounding each factor to a nearby multiple of 10, and compare this big part with the exact product. Then draw a rectangle for the correction factor, estimate its area, and correct the big part.

5.2.2 Low-entropy expressions

The correction to 3.15×7.21 was complicated as an absolute or additive change but simple as a fractional change. This contrast is general. Using the additive correction, a two-factor product becomes

$$(x + \Delta x)(y + \Delta y) = xy + \underbrace{x\Delta y + y\Delta x + \Delta x\Delta y}_{\text{additive correction}}. \tag{5.5}$$

Problem 5.8 Rectangle picture
Draw a rectangle representing the expansion

$$(x + \Delta x)(y + \Delta y) = xy + x\Delta y + y\Delta x + \Delta x\Delta y. \tag{5.6}$$

When the absolute changes Δx and Δy are small ($x \ll \Delta x$ and $y \ll \Delta y$), the correction simplifies to $x\Delta y + y\Delta x$, but even so it is hard to remember because it has many plausible but incorrect alternatives. For example, it could plausibly contain terms such as $\Delta x\Delta y$, $x\Delta x$, or $y\Delta y$. The extent of the plausible alternatives measures the gap between our intuition and reality; the larger the gap, the harder the correct result must work to fill it, and the harder we must work to remember the correct result.

Such gaps are the subject of statistical mechanics and information theory [20, 21], which define the gap as the logarithm of the number of plausible alternatives and call the logarithmic quantity the entropy. The logarithm does not alter the essential point that expressions differ in the number of plausible alternatives and that high-entropy expressions [28]—ones with many plausible alternatives—are hard to remember and understand.

In contrast, a low-entropy expression allows few plausible alternatives, and elicits, "Yes! How could it be otherwise?!" Much mathematical and scientific progress consists of finding ways of thinking that turn high-entropy expressions into easy-to-understand, low-entropy expressions.

▶ *What is a low-entropy expression for the correction to the product xy?*

A multiplicative correction, being dimensionless, automatically has lower entropy than the additive correction: The set of plausible dimensionless expressions is much smaller than the full set of plausible expressions.

The multiplicative correction is $(x + \Delta x)(y + \Delta y)/xy$. As written, this ratio contains gratuitous entropy. It constructs two dimensioned sums $x+\Delta x$ and $y+\Delta y$, multiplies them, and finally divides the product by xy. Although the result is dimensionless, it becomes so only in the last step. A cleaner method is to group related factors by making dimensionless quantities right away:

$$\frac{(x+\Delta x)(y+\Delta y)}{xy} = \frac{x+\Delta x}{x}\,\frac{y+\Delta y}{y} = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right). \tag{5.7}$$

The right side is built only from the fundamental dimensionless quantity 1 and from meaningful dimensionless ratios: $(\Delta x)/x$ is the fractional change in x, and $(\Delta y)/y$ is the fractional change in y.

The gratuitous entropy came from mixing $x + \Delta x$, $y + \Delta y$, x, and y willy nilly, and it was removed by regrouping or unmixing. Unmixing is difficult with physical systems. Try, for example, to remove a drop of food coloring mixed into a glass of water. The problem is that a glass of water contains roughly 10^{25} molecules. Fortunately, most mathematical expressions have fewer constituents. We can often regroup and unmix the mingled pieces and thereby reduce the entropy of the expression.

Problem 5.9 Rectangle for the correction factor

Draw a rectangle representing the low-entropy correction factor

$$\left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right). \tag{5.8}$$

A low-entropy correction factor produces a low-entropy fractional change:

$$\frac{\Delta\,(xy)}{xy} = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right) - 1 = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta x}{x}\frac{\Delta y}{y}, \tag{5.9}$$

where $\Delta(xy)/xy$ is the fractional change from xy to $(x+\Delta x)(y+\Delta y)$. The rightmost term is the product of two small fractions, so it is small compared to the preceding two terms. Without this small, quadratic term,

$$\frac{\Delta\,(xy)}{xy} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}. \tag{5.10}$$

Small fractional changes simply add!

This fractional-change rule is far simpler than the corresponding approximate rule that the absolute change is $x\Delta y + y\Delta x$. Simplicity indicates low entropy; indeed, the only plausible alternative to the proposed rule is the possibility that fractional changes multiply. And this conjecture is not likely: When $\Delta y = 0$, it predicts that $\Delta(xy) = 0$ no matter the value of Δx (this prediction is explored also in Problem 5.12).

Problem 5.10 Thermal expansion

If, due to thermal expansion, a metal sheet expands in each dimension by 4%, what happens to its area?

Problem 5.11 Price rise with a discount

Imagine that inflation, or copyright law, increases the price of a book by 10% compared to last year. Fortunately, as a frequent book buyer, you start getting a store discount of 15%. What is the net price change that you see?

5.2.3 Squaring

In analyzing the engineered and natural worlds, a common operation is squaring—a special case of multiplication. Squared lengths are areas, and squared speeds are proportional to the drag on most objects (Section 2.4):

$$F_d \sim \rho v^2 A, \tag{5.11}$$