KELVIN'S SHIP WAVE PATTERN

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https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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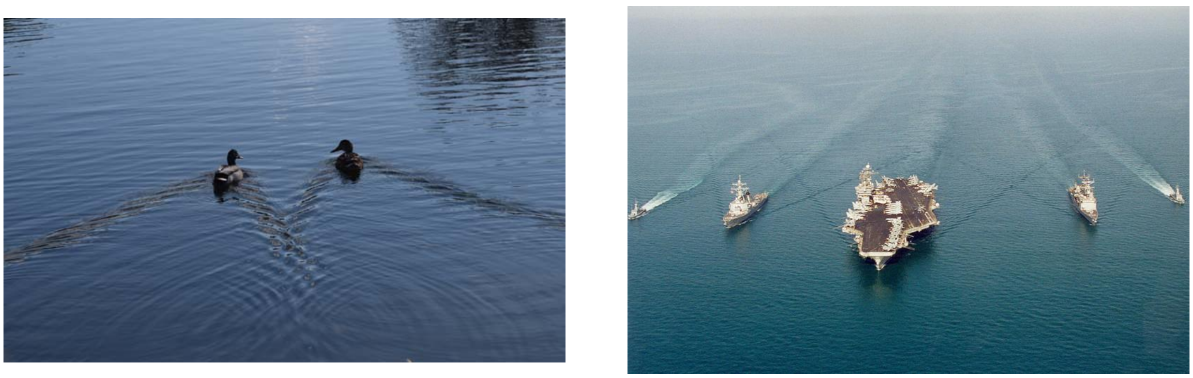

Figure 1: Examples of Kelvin's ship wave pattern

When a disturbance (e.g. a ship or a duck) travels on a water surface, it carries with it a familiar pattern of waves which was first explained by Lord Kelvin in 1891. The pattern is known today as the Kelvin wake pattern. Kelvin's theory assumed that the magnitude of the pressure and the resulting wave were small, so that the equations of motion could be linearized; the viscosity of water and the surface tension were neglected; the depth of the water was assumed to be large relative to the characteristic wavelength of the pattern.

The properties of a ship's wake follow entirely from the dispersion relation for deep-water waves. In still water, a small-amplitude periodic disturbance the the surface with horizontal wave number $\mathbf{k} = k_x\hat{x} + k_y\hat{y}$ oscillates at angular frequency

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|}. \tag{1}$$

where g is the gravitational acceleration. (Recall that the wavelength $\lambda = \frac{2\pi}{|\mathbf{k}|}$.)

The form of this relation can be justified by dimensional analysis once it is recognized that the only relevant quantities are g, the magnitude of the wave-vector, and possibly the density of the water.

The change due to waves in the water height, $z(\mathbf{x}, t)$, at a point $\mathbf{x}$ at time t can be written as a sum of all possible waves

$$z(\mathbf{r}, t) = \int A(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, \mathrm{d}^2\mathbf{k}, \tag{2}$$

for some unspecified yet amplitude $A(\mathbf{k})$.

We now place ourselves in the reference frame of a ship moving over the still water at velocity $\mathbf{u}$. In this frame, the water appears to be moving at velocity $-\mathbf{u}$, and the dispersion relations are Doppler shifted to

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|} - \mathbf{u}\cdot\mathbf{k}. \tag{3}$$

Asking for stationary waves in the ship frame with $\omega(\mathbf{k}) = 0$, we reduce the wave vector integration in Eq. (2) to one dimensional integration (i.e. by an angle) by using the relation

$$\mathbf{k} = \frac{g}{(\mathbf{u}\cdot\hat{\mathbf{k}})^2}\,\hat{\mathbf{k}}, \tag{4}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$,

$$\hat{\mathbf{k}} = (\cos\theta, \sin\theta). \tag{5}$$

The angle θ is measured with respect to the direction of $\mathbf{u}$:

$$\mathbf{u}\cdot\hat{\mathbf{k}} = u\cos\theta. \tag{6}$$

The height is then given by

$$z(\mathbf{r}) = \int_0^{2\pi} A(\theta)\exp\left[ig\frac{\mathbf{r}\cdot\hat{\mathbf{k}}}{(\mathbf{u}\cdot\hat{\mathbf{k}})^2}\right]\mathrm{d}\theta. \tag{7}$$

The information about the ship is encoded in the amplitudes $A(\theta_k)$. Let assume that a point ship radiating backwards uniformly in all directions: A is constant over $\theta \in (-\pi/2, \pi/2)$, and zero outside it. Rewriting the integral in polar coordinates, r and ϕ,

$$\mathbf{r} = r(\cos\phi, -\sin\phi), \tag{8}$$

and introducing the dimensionless distance $\rho = \frac{g}{u^2}r$, we obtain

$$z(\phi,\rho) = \int_{-\pi/2}^{\pi/2} \cos\left(\rho\,\frac{\cos(\theta+\phi)}{\cos^2\theta}\right)\mathrm{d}\theta. \tag{9}$$

The wake pattern calculated using the integral Eq. (9) is presented in Fig. (2).

The section on *Integrals with Coalescing Saddles*, the Digital Library of Mathematical Functions (DLMF) provides the analysis of the Kelvin wake pattern, starting from Equation (9).

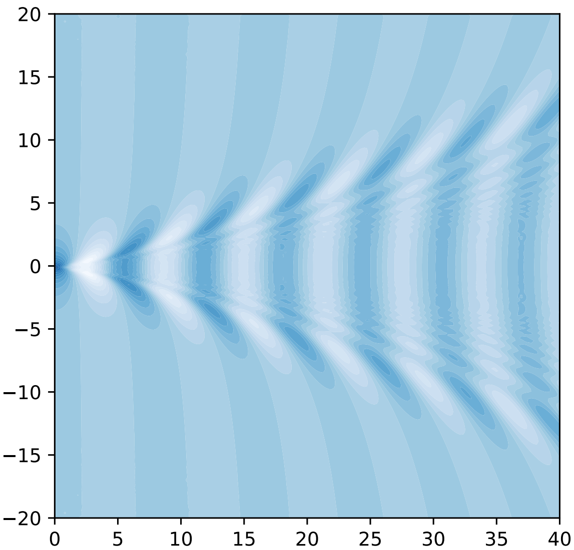

Figure 2: Kelvin wake calculated using Eq. (9)

General references

[1] Fritz Ursell. "On Kelvin's ship-wave pattern". In: *Journal of Fluid Mechanics* 8.3 (1960), pp. 418–431.

[2] *NIST Digital Library of Mathematical Functions, §36.13 Kelvin's Ship-Wave Pattern*. F. W. J. Olver, et al. eds. URL: https://dlmf.nist.gov/36.13.

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