JORDAN'S LEMMA

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https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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When evaluating integrals using complex variables, we often need to show that

$$\lim_{R\to\infty} \int_{C_R} g(z)\, dz = 0, \tag{1}$$

where the integration contour C_R is a semicircular arc of radius R in the upper half-plane, $z = Re^{i\theta}$, $0 \le \theta \le \pi$, and $g(z)$ is an analytic function (except possibly for a finite number of poles).

This is true if $g(z)$ decreases $\sim \frac{1}{|z|^2}$ (or faster) as $z \to \infty$. Indeed, on a circular arc

$$\mathrm{d}z = i\, R\, e^{i\theta}\, \mathrm{d}\theta, \quad 0 \le \theta \le \pi. \tag{2}$$

The absolute value of integral Eq. (1),

$$\left| \int_{C_R} g(z)\, dz \right| = \left| iR \int_0^\pi g\left(Re^{i\theta}\right) \mathrm{d}\theta \right| \le R \int_0^\pi \left| g\left(Re^{i\theta}\right)\right| \mathrm{d}\theta \tag{3}$$

$$\le R \frac{m}{R^2} \int_0^\pi \mathrm{d}\theta \sim \frac{1}{R}. \tag{4}$$

Therefore, the absolute value of the integral and hence the integral itself are 0 as $R \to \infty$.

Jordan's Lemma extends this result for a special form of $g(z)$,

$$g(z) = f(z)\, e^{i\lambda z}, \quad \lambda > 0, \tag{5}$$

where is a real parameter, $\lambda > 0$, from functions $F(z)$ satisfying $f(z) \sim \frac{1}{|z|^2}$ as $|z| \to \infty$ to functions $f(z)$ satisfying $f(z) \to 0$ as $|z| \to \infty$. For $\lambda < 0$, the same conclusion holds for the semicircular contour C_R in the lower half-plane.

Indeed,

$$\int\limits_{C_R} f(z)\, e^{i\lambda z}\, dz = iR \int_0^{\pi} f\left(Re^{i\theta}\right) e^{i(\lambda R\cos(\theta)+\theta)}\, e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta. \tag{6}$$

Therefore,

$$\begin{aligned}
\left| \int\limits_{C_R} f(z)\, e^{i\lambda z}\, dz \right| &= \left| iR \int_0^{\pi} f\left(Re^{i\theta}\right) e^{i(\lambda R\cos(\theta)+\theta)}\, e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta \right| &(7)\\
&\le R \int_0^{\pi} \left| f\left(Re^{i\theta}\right) e^{i(\lambda R\cos(\theta)+\theta)}\, e^{-\lambda R\sin(\theta)} \right| \mathrm{d}\theta &(8)\\
&= R \int_0^{\pi} \left| f\left(Re^{i\theta}\right) \right| e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta &(9)\\
&\le R\, M(R) \int_0^{\pi} e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta \le R\, M(R)\, \frac{\pi}{\lambda R} = \frac{\pi\, M(R)}{\lambda}. &(10)
\end{aligned}$$

Here

$$M(R) = \max_{0 \le \theta \le \pi} \left| f\left(Re^{i\theta}\right) \right|, \tag{11}$$

and we used that (see Fig. 1)

$$\begin{aligned}
\int\limits_0^{\pi} e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta &= 2\int\limits_0^{\frac{\pi}{2}} e^{-\lambda R\sin(\theta)}\, \mathrm{d}\theta \le 2\int\limits_0^{\frac{\pi}{2}} e^{-\lambda R\frac{2\theta}{\pi}}\, \mathrm{d}\theta = \frac{\pi}{\lambda R}\int\limits_0^{\frac{\pi}{2}} e^{-\frac{2\lambda R\theta}{\pi}}\, \mathrm{d}\left(\frac{2\lambda R\theta}{\pi}\right) &(12)\\
&= \frac{\pi}{\lambda R}\int\limits_0^{\lambda R} e^{-u}\, \mathrm{d}u = \frac{\pi}{\lambda R}\left(1 - e^{-\lambda R}\right) \le \frac{\pi}{\lambda R}. &(13)
\end{aligned}$$

Since $M(R) \to 0$ as $R \to \infty$, the absolute value of integral Eq. (6) and hence the integral itself are 0 as $R \to \infty$.

Figure 1: Illustration of the inequality $\sin(\theta) \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$.