## JORDAN'S LEMMA

**Spring 2023** 

https://www.phys.uconn.edu/~rozman/Courses/P2400\_23S/

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When evaluating integrals using complex variables, we often need to show that

$$\lim_{R \to \infty} \int_{C_R} g(z) dz = 0, \tag{1}$$

where the integration contour  $C_R$  is a semicircular arc of radius R in the upper half-plane,  $z = Re^{i\theta}$ ,  $0 \le \theta \le \pi$ , and g(z) is an analytic function (except possibly for a finite number of poles).

This is true if g(z) decreases  $\sim \frac{1}{|z|^2}$  (or faster) as  $z \to \infty$ . Indeed, on a circular arc

$$dz = i R e^{i\theta} d\theta, \quad 0 \le \theta \le \pi. \tag{2}$$

The absolute value of integral Eq. (1),

$$\left| \int_{C_R} g(z) dz \right| = \left| iR \int_0^{\pi} g\left(Re^{i\theta}\right) d\theta \right| \le R \int_0^{\pi} \left| g\left(Re^{i\theta}\right) \right| d\theta \tag{3}$$

$$\leq R \frac{m}{R^2} \int_0^{\pi} d\theta \sim \frac{1}{R}.$$
 (4)

Therefore, the absolute value of the integral and hence the integral itself are 0 as  $R \to \infty$ . Jordan's Lemma extends this result for a special form of g(z),

$$g(z) = f(z)e^{i\lambda z}, \quad \lambda > 0,$$
 (5)

where is a real parameter,  $\lambda > 0$ , from functions F(z) satisfying  $f(z) \sim \frac{1}{|z|^2}$  as  $|z| \to \infty$  to functions f(z) satisfying  $f(z) \to 0$  as  $|z| \to \infty$ . For  $\lambda < 0$ , the same conclusion holds for the semicircular contour  $C_R$  in the lower half-plane.

Indeed,

$$\int_{C_R} f(z) e^{i\lambda z} dz = iR \int_0^{\pi} f\left(Re^{i\theta}\right) e^{i(\lambda R\cos(\theta) + \theta)} e^{-\lambda R\sin(\theta)} d\theta. \tag{6}$$

Therefore,

$$\left| \int_{C_R} f(z) e^{i\lambda z} dz \right| = \left| iR \int_0^{\pi} f\left(Re^{i\theta}\right) e^{i(\lambda R\cos(\theta) + \theta)} e^{-\lambda R\sin(\theta)} d\theta \right|$$
 (7)

$$\leq R \int_{0}^{\pi} \left| f(Re^{i\theta}) e^{i(\lambda R\cos(\theta) + \theta)} e^{-\lambda R\sin(\theta)} \right| d\theta \tag{8}$$

$$= R \int_0^{\pi} \left| f\left(Re^{i\theta}\right) \right| e^{-\lambda R \sin(\theta)} d\theta \tag{9}$$

$$\leq RM(R) \int_0^{\pi} e^{-\lambda R \sin(\theta)} d\theta \leq RM(R) \frac{\pi}{\lambda R} = \frac{\pi M(R)}{\lambda}. \tag{10}$$

Here

$$M(R) = \max_{0 \le \theta \le \pi} \left| f\left(Re^{i\theta}\right) \right|,\tag{11}$$

and we used that (see Fig. 1)

$$\int_{0}^{\pi} e^{-\lambda R \sin(\theta)} d\theta = 2 \int_{0}^{\frac{\pi}{2}} e^{-\lambda R \sin(\theta)} d\theta \le 2 \int_{0}^{\frac{\pi}{2}} e^{-\lambda R \frac{2\theta}{\pi}} d\theta = \frac{\pi}{\lambda R} \int_{0}^{\frac{\pi}{2}} e^{-\frac{2\lambda R\theta}{\pi}} d\left(\frac{2\lambda R\theta}{\pi}\right)$$
(12)

$$= \frac{\pi}{\lambda R} \int_{0}^{\Lambda R} e^{-u} du = \frac{\pi}{\lambda R} \left( 1 - e^{-\lambda R} \right) \le \frac{\pi}{\lambda R}.$$
 (13)

Since  $M(R) \to 0$  as  $R \to \infty$ , the absolute value of integral Eq. (6) and hence the integral itself are 0 as  $R \to \infty$ .

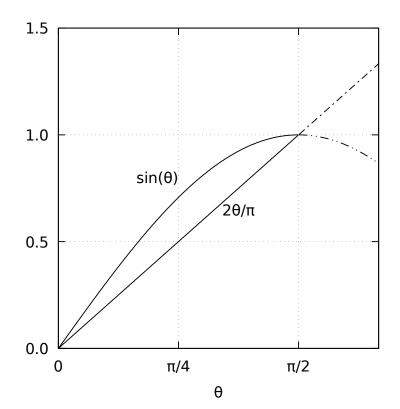


Figure 1: Illustration of the inequality  $\sin(\theta) \ge \frac{2\theta}{\pi}$  for  $0 \le \theta \le \frac{\pi}{2}$ .