Name: ______________________________

Date: ______________________________

Collaborators: ____

Question:	1	Total
Points:	65	65
Score:		

Instructor/grader comments:

Laplace method for ODEs

1. Use the Laplace's method for differential equations to solve the following initial value problem for $x \geq 0$:
$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \ .$$

   (a) (15 points) Use the Laplace's method to find the integral solution containing one integration constant. Use a CCW complex closed contour around the origin as the integration contour.

   (b) (15 points) Use your integral representation of $y(x)$, to calculate $y(0)$ and $y'(0)$. Chose the integration constant to satisfy the initial conditions.

   Hint: Recall that
$$e^{-\frac{1}{z}} = 1 - \frac{1}{1!\,z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \dots$$
   Construct the Laurent series for the integrands for $y(0)$ and $y'(0)$.

   (c) (10 points) Show that the integral you obtained is indeed the solution of the ordinary differential equation above.

   Hint: Differentiate twice under the integral sign, multiply by x, and integrate by parts once.

   (d) (10 points) Let t be the integration variable in your expression for $y(x)$. Introduce a new integration variable, $u = \sqrt{x}t$ and rewrite the integral in "symmetric" form. Deform the integration contour to a unit circle in the complex plane. Make another change of integration variable, $u = e^{i\phi}$, $0 \leq \phi \leq 2\pi$. Since your solution must be a real function of x, replace the integrand with its real part.

   Hint: recall that $\cos(\alpha) = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$, $\sin(\alpha) = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$, $\mathrm{Re}\left(e^{i\alpha}\right) = \cos(\alpha)$.

   (e) (15 points) To verify your solution, plot on the same graph your integral solution (in the form obtained in Part d) and the solution of the original initial value problem. Use a computer algebra system. Attach a printout of your CAS session.

   Hint: For Mathematica, you may use the following code:

```
ode = {x * y''[x] + y[x] == 0, y[0] == 0, y'[0] == 1}
sol = DSolve[ode, y, {x, 0, 10}]
lapsol[x_] = ... * Integrate[your integrand here, {ϕ, 0, 2 * Pi}]
Plot[{lapsol[x], y[x]/.sol}, {x, 0, 10}]
```

The expected graph is shown in Fig. 1.

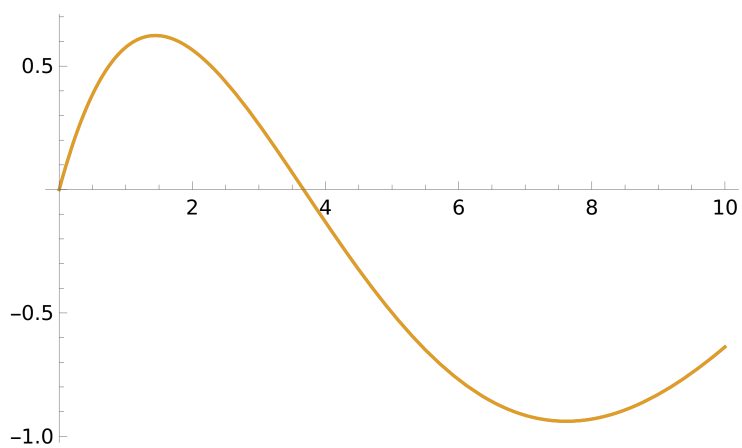

Figure 1: Expected result in Problem 1: two graphs overlap.