Name: ______________________________

Date: ______________________________

Collaborators: ____

Question:	1	2	3	4	5	Total
Points:	30	15	10	10	10	75
Score:						

Instructor/grader comments:

Integral multiplication trick: the Fresnel integrals

1. (30 points) Evaluate the following integrals

$$C = \int_0^\infty \cos\left(x^2\right) \mathrm{d}x, \quad \text{and} \quad S = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x.$$

The integrals C and S are named after the Fresnel (French physicist, 1788-1827). Despite the name, they were first evaluated by Euler (in 1781).

Hints: use Euler formula to write the integral for $F = C + iS$. Square the integral (recall that if $I = \int_a^b f(x)\,\mathrm{d}x$ then $I^2 = \iint_a^b f(x) f(y)\,\mathrm{d}x\mathrm{d}y$). Evaluate the double integral in polar coordinates. Notice that $\int_0^\infty g(r^2)\, r\,\mathrm{d}r = \frac{1}{2}\int_0^\infty g(r^2)\,\mathrm{d}r^2 = \frac{1}{2}\int_0^\infty g(t)\,\mathrm{d}t$ for an arbitrary integrand $g()$. Temporary add a convergence factor in the form $e^{-\lambda r^2}$ where $\lambda \to +0$.

Answer: $C = \sqrt{\frac{\pi}{8}}$, $S = \sqrt{\frac{\pi}{8}}$

Gamma function

2. (15 points) Evaluate the integral in terms of Gamma function. Simplify the expression as much as possible.

$$I = \int_0^\infty e^{-x^4} \mathrm{d}x$$

Answer: $I = \Gamma\left(\frac{5}{4}\right)$

Complex numbers

3. (10 points) Find the coordinate and the polar form of the following complex number:

$$Z = \left(\frac{\sqrt{2} - i\sqrt{2}}{1 - i\sqrt{3}}\right)^{26}.$$

Answer: $Z = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

4. (10 points) Find the values of $Z = \left(\sqrt{i}\right)^i$.

Answer: $Z = e^{-\frac{\pi}{4} - \pi n}$, where $n = 0, \pm 1, \pm 2, \ldots$

5. (10 points) Find the coordinate and the polar forms of the solutions of the equation:

$$z^4 = \sqrt{3} - i.$$

How many roots are there?