EVALUATE GAUSSIAN AND FRESNEL'S INTEGRALS USING DIFFERENTIATION UNDER THE INTEGRAL SIGN

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In this note we evaluate Gaussian integral,

$$I_g \equiv \int_0^\infty e^{-x^2} \,\mathrm{d}x = \frac{\sqrt{\pi}}{2},\tag{1}$$

and Fresnel integrals,

$$I_c \equiv \int_0^\infty \cos\left(x^2\right) \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}} \tag{2}$$

and

$$I_s \equiv \int_0^\infty \sin\left(x^2\right) dx = \frac{\sqrt{\pi}}{2\sqrt{2}},\tag{3}$$

using differentiation under the integral sign.

The two parts of this note can be read independently.

Gaussian integral

Consider the following integral

$$J(x) = \int_{0}^{\infty} \frac{e^{-x^{2}(1+y^{2})}}{1+y^{2}} \,\mathrm{d}y.$$
(4)

We know its values for x = 0:

$$J(0) = \int_{0}^{\infty} \frac{dy}{1 + y^{2}} = \arctan(\infty) = \frac{\pi}{2},$$
(5)

and for $x = \infty$:

$$J(\infty) = 0. \tag{6}$$

The derivative of J(x),

$$\frac{dJ}{dx} = -2x \int_{0}^{\infty} e^{-x^{2}(1+y^{2})} dy = -2e^{-x^{2}} \int_{0}^{\infty} e^{-(xy)^{2}} d(xy)$$
$$= -2e^{-x^{2}} \int_{0}^{\infty} e^{-u^{2}} du$$
$$= -2e^{-x^{2}} I_{g}.$$
(7)

Integrating Eq. (7) with respect to *x* from 0 to ∞ , we obtain:

$$J(\infty) - J(0) = -2I_g \int_0^\infty e^{-x^2} dx = -2I_g^2,$$
 (8)

or, using Eq. (5) and Eq. (6),

$$I_g^2 = \frac{J(0)}{2} = \frac{\pi}{4},\tag{9}$$

i.e.

$$I_g = \frac{\sqrt{\pi}}{2}.$$
 (10)

Fresnel's integrals

Consider the following integrals:

$$c(x) = \int_{0}^{\infty} \frac{\cos\left(x^{2}(1+y^{2})\right)}{1+y^{2}} \,\mathrm{d}y, \tag{11}$$

$$s(x) = \int_{0}^{\infty} \frac{\sin\left(x^{2}(1+y^{2})\right)}{1+y^{2}} \,\mathrm{d}y.$$
(12)

We know their values for x = 0:

$$c(0) = \int_{0}^{\infty} \frac{dy}{1 + y^{2}} = \arctan(\infty) = \frac{\pi}{2},$$
(13)

$$s(0) = 0,$$
 (14)

and for $x = \infty$:

$$c(\infty) = s(\infty) = 0. \tag{15}$$

The derivative of c(x),

$$\frac{\mathrm{d}c}{\mathrm{d}x} = -2x \int_{0}^{\infty} \sin\left(x^{2}(1+y^{2})\right) \mathrm{d}y = -2x \int_{0}^{\infty} \sin\left(x^{2}+(xy)^{2}\right) \mathrm{d}y$$

$$= -2\sin(x^{2}) \int_{0}^{\infty} \cos(xy)^{2} \mathrm{d}(xy) - 2\cos(x^{2}) \int_{0}^{\infty} \sin(xy)^{2} \mathrm{d}(xy)$$

$$= -2\sin(x^{2}) \int_{0}^{\infty} \cos(u)^{2} \mathrm{d}u - 2\cos(x^{2}) \int_{0}^{\infty} \sin(u^{2}) \mathrm{d}u$$

$$= -2\sin(x^{2}) I_{c} - 2\cos(x^{2}) I_{s}.$$
(16)

Similarly,

$$\frac{ds}{dx} = 2\cos(x^2)I_c - 2\sin(x^2)I_s.$$
(17)

Integrating Eqs. (16), (17) with respect to *x* from 0 to ∞ , we obtain:

$$c(\infty) - c(0) = -2I_c \int_0^\infty \sin(x^2) \, \mathrm{d}x - 2I_s \int_0^\infty \cos(x^2) \, \mathrm{d}x = -4I_c I_s, \tag{18}$$

$$s(\infty) - s(0) = 2I_c \int_0^\infty \cos(x^2) \, \mathrm{d}x - 2I_s \int_0^\infty \sin(x^2) \, \mathrm{d}x = 2I_c^2 - 2I_s^2.$$
(19)

Using the boundary conditions Eqs. (13)-(15), $c(\infty) - c(0) = -\frac{\pi}{2}$, $s(\infty) - s(0) = 0$, therefore

$$I_c I_s = \frac{\pi}{8}, \quad I_c = I_s. \tag{20}$$

Finally,

$$I_c = I_s = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$
(21)

References

[1] J. van Yzeren, "Moivre's and Fresnel's integrals by simple integration," *The American Mathematical Monthly*, vol. 86, no. 8, pp. 690–693, 1979.