

EULER'S FORMULA

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https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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1 Introduction

In complex analysis Euler's formula establishes the relation between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number θ :

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad (1)$$

where e is the base of the natural logarithm, i is the imaginary unit, $i = \sqrt{-1}$, and $\cos \theta$ and $\sin \theta$ are the trigonometric functions, with the argument θ given in radians. The formula Eq. (1) was published by Euler in 1748, obtained by comparing the series expansions of the exponential and trigonometric expressions.

2 Derivation

Consider

$$z(\theta) = \cos \theta + i \sin \theta, \quad (2)$$

where θ is a real parameter.

Let's take the derivative of z with respect to θ :

$$\frac{dz}{d\theta} = -\sin \theta + i \cos \theta = i(\cos \theta + i \sin \theta) = iz. \quad (3)$$

Eq. (3) is a first order ordinary differential equation. Separating the variables and integrating, we obtain:

$$\frac{dz}{z} = i d\theta, \quad (4)$$

$$\ln z = i\theta + C', \quad (5)$$

where C' is an integration constant.

Exponentiating Eq. (5), we get:

$$z(\theta) = Ce^{i\theta}, \quad (6)$$

where $C = e^{C'}$ is just another constant.

From Eq. (2),

$$z(0) = 1. \quad (7)$$

From Eq. (6)

$$z(0) = C. \quad (8)$$

Therefore, $C = 1$ and

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (9)$$

For $\theta = \pi$ Eq. (9) establishes the relation between the three fundamental constants — e , π , and i :

$$\boxed{e^{i\pi} = -1}. \quad (10)$$

3 Geometric interpretation and the complex plane

A point in the complex plane can be represented by a complex number written in cartesian coordinates. Euler's formula provides a means of conversion between cartesian coordinates and polar coordinates. The polar form simplifies the mathematics when used in multiplication, division, or powers of complex numbers.

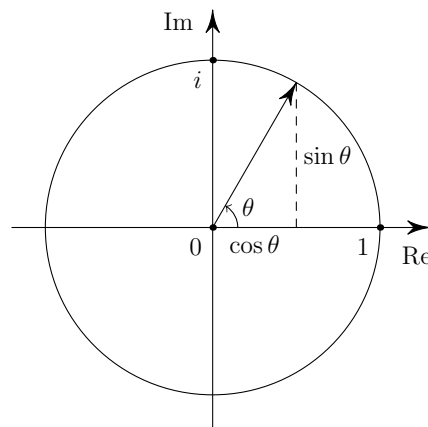


Figure 1: Euler's formula illustrated in the complex plane.