DIFFERENTIATING UNDER THE INTEGRAL SIGN

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https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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1 The case of the integrand depending on a parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b f(x,t)\,\mathrm{d}x\right) = \int_a^b \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x, \tag{1}$$

where

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x, \tag{2}$$

a, b are fixed parameters.

Indeed, considering the definition of the derivative as the limit,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t\to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t}, \tag{3}$$

and expanding $f(x, t+\Delta t)$ into Taylor series,

$$f(x,t+\Delta t) = f(x,t) + \frac{\partial f}{\partial t}\Delta t + O\left(\Delta t^2\right). \tag{4}$$

$$\begin{aligned} I(t+\Delta t) - I(t) &= \int_a^b f(x,t+\Delta t)\,\mathrm{d}x - \int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \Big[f(x,t+\Delta t) - f(x,t)\Big]\,\mathrm{d}x \\ &= \int_a^b \left[\frac{\partial f}{\partial t}\Delta t + O\Big(\Delta t^2\Big)\right]\mathrm{d}x = \left[\int_a^b \left(\frac{\partial f}{\partial t}\right)\mathrm{d}x\right]\Delta t + O\Big(\Delta t^2\Big). \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (3), and taking the limit $\Delta t \to 0$, we obtain Eq. (1).

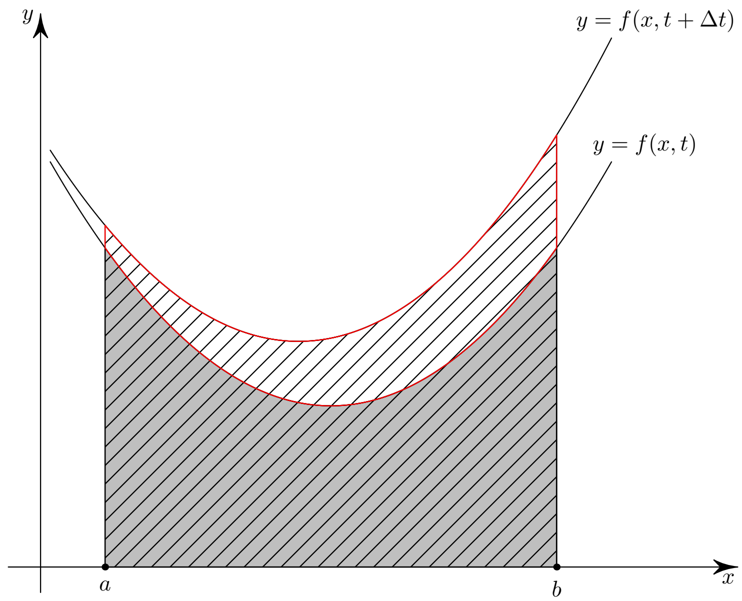

Figure 1: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (5).

2 Case of the integration range depending on a parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x\right) = f\Big(b(t)\Big)\frac{\mathrm{d}b}{\mathrm{d}t} - f\Big(a(t)\Big)\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{6}$$

where

$$I(t) = \int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x. \tag{7}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} \tag{8}$$

$$I(t+\Delta t) = \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x. \tag{9}$$

$$\begin{aligned} I(t+\Delta t) - I(t) &= \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x = \int_{b(t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int_{a(t)}^{a(t+\Delta t)} f(x)\,\mathrm{d}x \\ &= \Big(b(t+\Delta t) - b(t)\Big) f\Big(b(t)\Big) - \Big(a(t+\Delta t) - a(t)\Big) f\Big(a(t)\Big) + O\Big(\Delta t^2\Big) \\ &= \left[f\Big(b(t)\Big)\frac{\mathrm{d}b}{\mathrm{d}t} - f\Big(a(t)\Big)\frac{\mathrm{d}a}{\mathrm{d}t}\right]\Delta t + O\Big(\Delta t^2\Big), \end{aligned} \tag{10}$$

where we used that $a(t+\Delta t) - a(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\Delta t + O\Big(\Delta t^2\Big)$ and similar for b.

Combining Eq. (10) and Eq. (8), and taking the limit $\Delta t \to 0$, we obtain Eq. (6).

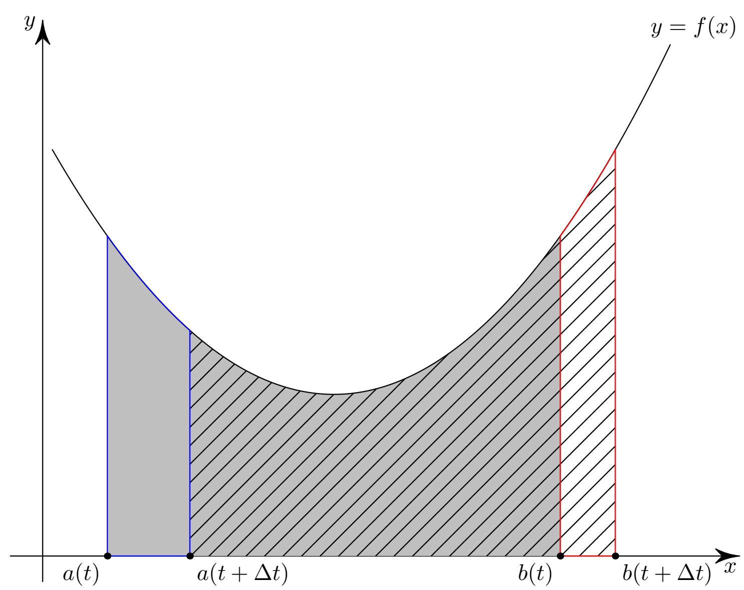

Figure 2: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (10).

3 General case

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x + f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{11}$$

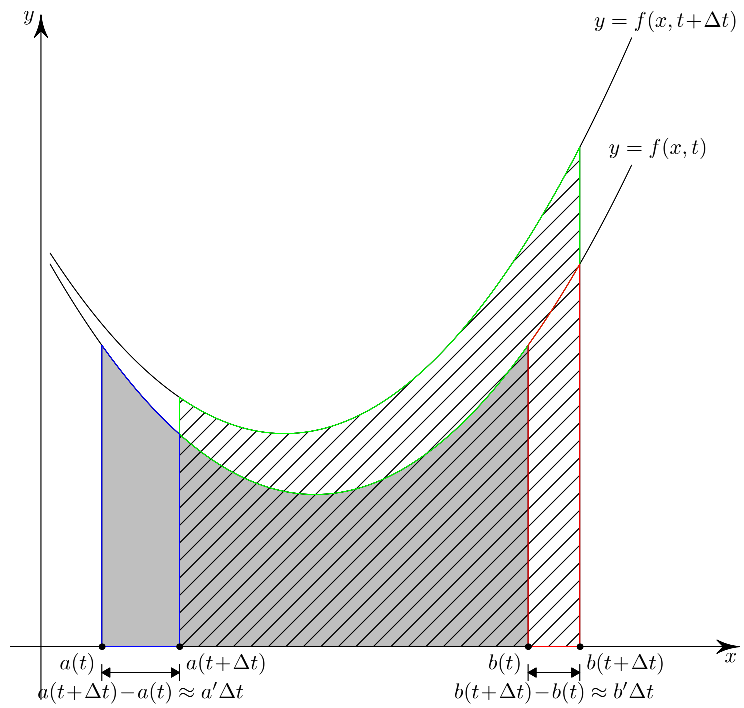

Figure 3: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference.