CAUCHY'S INTEGRAL THEOREM: EXAMPLES

Spring semester 2023

https://www.phys.uconn.edu/~rozman/Courses/P2400_23S/

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Cauchy's theorem states that if $f(z)$ is analytic at all points on and inside a closed complex contour C, then the integral of the function around that contour vanishes:

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{1}$$

1 A trigonometric integral

Problem: Show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha\phi)\,[\cos\phi]^{\alpha-1}\,\mathrm{d}\phi = 2^{\alpha}B(\alpha,\alpha) = 2^{\alpha}\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \tag{2}$$

Solution:

Recall the definition of Beta function,

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x, \tag{3}$$

and consider the integral:

$$J = \oint_C [z(1-z)]^{\alpha-1}\,\mathrm{d}z = 0, \quad \alpha > 1, \tag{4}$$

where the integration is over closed contour shown in Fig. 1.

Since the integrand in Eq. (4) is analytic inside C,

$$J = 0. \tag{5}$$

On the other hand,

$$J = J_{\mathrm{I}} + J_{\mathrm{II}}, \tag{6}$$

where J_{I} is the integral along the segment of the positive real axis, $0 \le x \le 1$; J_{II} is the integral along the circular arc or radius $R = \frac{1}{2}$ centered at $z = \frac{1}{2}$.

Along the real axis $z = x$, $\mathrm{d}z = \mathrm{d}x$, thus

$$J_{\mathrm{I}} = \int_0^1 x^{\alpha-1}\,(1-x)^{\alpha-1}\,\mathrm{d}x = B(\alpha, \alpha). \tag{7}$$

Along the semi-circular arc

$$z = \frac{1}{2} + \frac{1}{2}\,e^{i\theta} = \frac{1}{2}\,e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right) = e^{i\frac{\theta}{2}}\cos\frac{\theta}{2}, \tag{8}$$

where $0 < \theta < \pi$, and

$$1 - z = \frac{1}{2} - \frac{1}{2}e^{i\theta} = -i\,e^{i\frac{\theta}{2}}\sin\frac{\theta}{2}. \tag{9}$$

Hence,

$$z(1-z) = -i\,e^{i\theta}\cos\frac{\theta}{2}\sin\frac{\theta}{2} = -\frac{i}{2}\,e^{i\theta}\sin\theta. \tag{10}$$

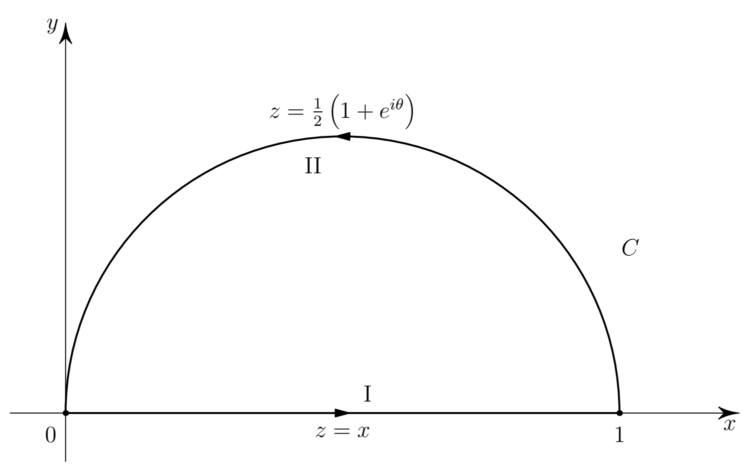

Figure 1: Integration contour for Problem 1

$$\mathrm{d}z = \frac{i}{2}\, e^{i\theta}\, \mathrm{d}\theta. \tag{11}$$

Therefore,

$$J_{\mathrm{II}} = -\left(\frac{-i}{2}\right)^{\alpha} \int_0^{\pi} e^{i\alpha\theta} (\sin\theta)^{\alpha-1}\, \mathrm{d}\theta = -2^{-\alpha} e^{-i\frac{\pi}{2}\alpha} \int_0^{\pi} e^{i\alpha\theta} (\sin\theta)^{\alpha-1}\, \mathrm{d}\theta, \tag{12}$$

where we used that

$$-i = e^{-i\frac{\pi}{2}}. \tag{13}$$

Combining Eqs. (5), (6), (7), and (12) we obtain:

$$2^{-\alpha} e^{-i\frac{\pi}{2}\alpha} \int_0^{\pi} e^{i\alpha\theta} (\sin\theta)^{\alpha-1}\, \mathrm{d}\theta = B(\alpha,\alpha). \tag{14}$$

Transforming the expression on the left as following:

$$\int_0^{\pi} e^{i\alpha\left(\theta-\frac{\pi}{2}\right)} (\sin\theta)^{\alpha-1}\, \mathrm{d}\theta = \int_0^{\pi} e^{i\alpha\left(\theta-\frac{\pi}{2}\right)} \left[\cos\left(\theta-\frac{\pi}{2}\right)\right]^{\alpha-1} \mathrm{d}\theta \tag{15}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\phi} \left[\cos\phi\right]^{\alpha-1} \mathrm{d}\phi \tag{16}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha\phi) \left[\cos\phi\right]^{\alpha-1} \mathrm{d}\phi \tag{17}$$

finally obtain the relation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha\phi) \left[\cos\phi\right]^{\alpha-1} \mathrm{d}\phi = 2^{\alpha} B(\alpha,\alpha) = 2^{\alpha} \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \tag{18}$$

2 Euler's log-sine integral.

Problem: Show that

$$\int_0^\pi \log(\sin x)\,\mathrm{d}x = -\pi\log 2. \tag{19}$$

Integral Eq. (19), is called Euler's log-sine integral. It was first evaluated (by Euler) in 1769.

Solution:

We start by integrating the function $f(z)$,

$$f(z) = \log\left(1 - e^{2iz}\right), \tag{20}$$

along the rectangular contour C with the corners at 0, π, $\pi + iR$, iR, indented at the corners when necessary (see Fig. 2), and letting $R \to \infty$.

$$J = \oint_C \log\left(1 - e^{2iz}\right)\mathrm{d}z. \tag{21}$$

On the one hand, the integrand in Eq. (21) is an analytic function inside C, therefore

$$J = 0. \tag{22}$$

On the other hand,

$$J = J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}} + J_{\mathrm{IV}} + J_{\mathrm{V}} + J_{\mathrm{VI}}, \tag{23}$$

where the subscripts corresponds to integration contours labeled in Fig. 2.

Consider first the integrals J_{II} and J_{IV}. The integrand $f(z)$ is a periodic function with the period π,

$$f(z) = f(z+\pi). \tag{24}$$

Indeed,

$$f(z+\pi) = \log\left(1 - e^{2i(z+\pi)}\right) = \log\left(1 - e^{2iz}e^{2\pi i}\right) = \log\left(1 - e^{2iz}\right) = f(z). \tag{25}$$

Therefore in J_{II} and J_{IV} we have equal integrands but we are integrating in the opposite directions. Therefore,

$$J_{\mathrm{II}} = -J_{\mathrm{IV}}, \tag{26}$$

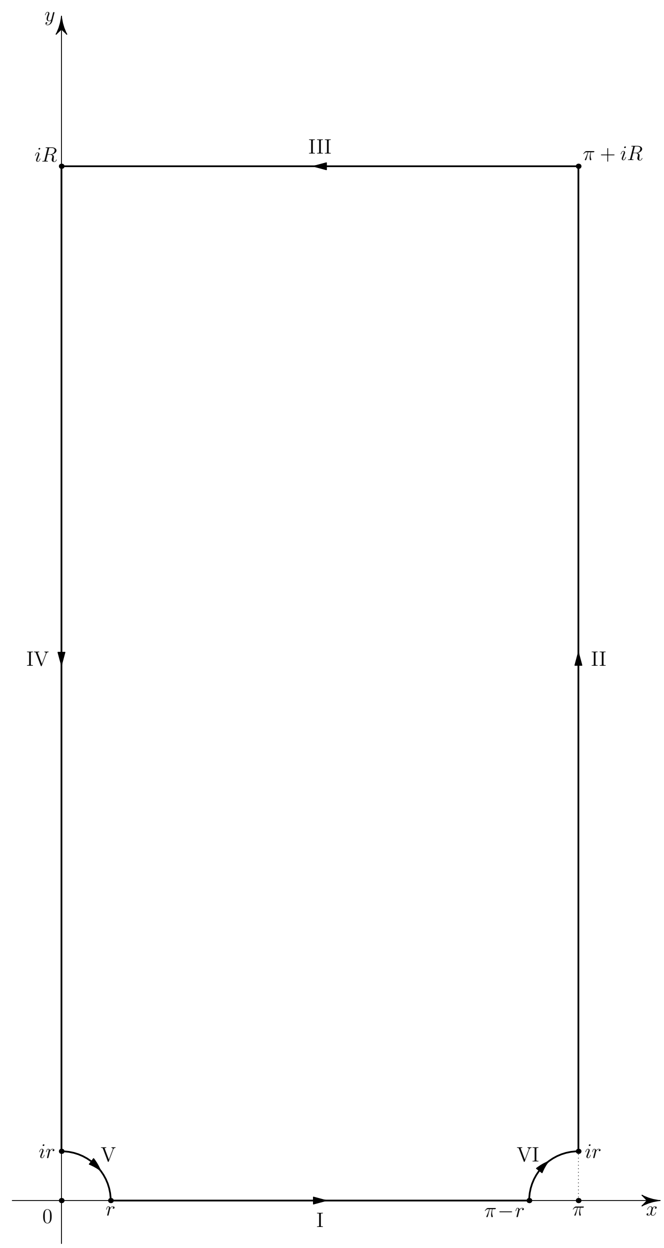

Figure 2: Integration contour for Problem 2

or

$$J_{\mathrm{II}} + J_{\mathrm{IV}} = 0. \tag{27}$$

Next, observe that $f(z) \to 0$ as $y = \mathrm{Im}(z) \to +\infty$:

$$f(z) = \log\left(1 - e^{2i(x+iy)}\right) = \log\left(1 - e^{2ix}e^{-y}\right) \approx -e^{2ix}e^{-y} \longrightarrow 0. \tag{28}$$

Therefore,

$$J_{\mathrm{III}} = 0. \tag{29}$$

Next, let's show that

$$J_{\mathrm{V}} \equiv \lim_{r\to 0} \int\limits_{C_{\mathrm{V}}} \log\left(1 - e^{2iz}\right) \mathrm{d}z = 0. \tag{30}$$

Indeed, $z = r\,e^{i\theta}$, $\mathrm{d}z = i\,r\,e^{i\theta}\mathrm{d}\theta$, $0 \le \theta \le \frac{\pi}{2}$:

$$J_{\mathrm{V}} = i \lim_{r\to 0} r \int\limits_0^{\frac{\pi}{2}} \log\left(1 - e^{2ire^{i\theta}}\right) e^{i\theta}\mathrm{d}\theta \approx i \lim_{r\to 0} r \int\limits_0^{\frac{\pi}{2}} \log\left(-2ire^{i\theta}\right) e^{i\theta}\mathrm{d}\theta = 0. \tag{31}$$

Similarly we can show that

$$J_{\mathrm{VI}} = 0. \tag{32}$$

Combining Eqs (22), (23), (27), (29), (30), and (32), we get that

$$J_{\mathrm{I}} = \int\limits_0^{\pi} \log\left(1 - e^{2ix}\right) \mathrm{d}x = 0. \tag{33}$$

Rewriting the integrand in Eq. (33) as following,

$$\begin{aligned} \log\left(1 - e^{2ix}\right) &= \log\left[e^{ix}\left(e^{-ix} - e^{ix}\right)\right] \\ &= \log\left[2(-i)\,e^{ix}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\right] = \log\left(2\,e^{i\left(x - \frac{\pi}{2}\right)} \sin x\right) \\ &= \log 2 + i\left(x - \frac{\pi}{2}\right) + \log(\sin x), \end{aligned} \tag{34}$$

we obtain

$$\int\limits_0^{\pi} \left[\log 2 + i\left(x - \frac{\pi}{2}\right) + \log(\sin x)\right] \mathrm{d}x = 0, \tag{35}$$

or

$$\int_0^{\pi} \log(\sin x)\,\mathrm{d}x = -\pi \log 2. \tag{36}$$

3 Another Euler integral

Problem: evaluate the following integral:

$$I(\alpha) = \int_0^{\infty} \frac{\sin(x)}{x^{\alpha}}\,\mathrm{d}x, \quad 0 < \alpha < 1. \tag{37}$$

Solution:

Let's consider the following integral:

$$J(\alpha) = \oint_C \frac{e^{iz}}{z^{\alpha}}\,\mathrm{d}z, \tag{38}$$

where the integration contour C is sketch in Fig. 3.

The integrand in Eq. (38) is an analytic function inside C, therefore

$$J(\alpha) = 0. \tag{39}$$

On the other hand,

$$J(\alpha) = J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}} + J_{\mathrm{IV}}. \tag{40}$$

where the subscripts corresponds to integration contours labeled in Fig. 3.

Let's consider J_{I}, J_{II}, J_{III}, and J_{IV} separately:

J_{I}**:** the integration is along the real axis, so $z = x$, $\mathrm{d}z = \mathrm{d}x$, $r \le x \le R$:

$$J_{\mathrm{I}} = \lim_{r\to 0} \lim_{R\to\infty} \int_r^R \frac{e^{ix}}{x^{\alpha}}\,\mathrm{d}x = \int_0^{\infty} \frac{e^{ix}}{x^{\alpha}}\,\mathrm{d}x, \tag{41}$$

so

$$I(\alpha) = \operatorname{Im} J_{\mathrm{I}}. \tag{42}$$

J_{II}: the integration is counterclockwise along the quarter-circle of radius R, $z = R\,e^{i\theta}$, $\mathrm{d}z = i\,R\,e^{i\theta}\mathrm{d}\theta$, $0 \le \theta \le \frac{\pi}{2}$:

$$J_{\mathrm{II}} = \lim_{R\to\infty} i\,R^{(1-\alpha)} \int_0^{\frac{\pi}{2}} e^{i\,R\cos\theta} e^{-R\sin\theta} e^{i(1-\alpha)\theta}\,\mathrm{d}\theta. \tag{43}$$

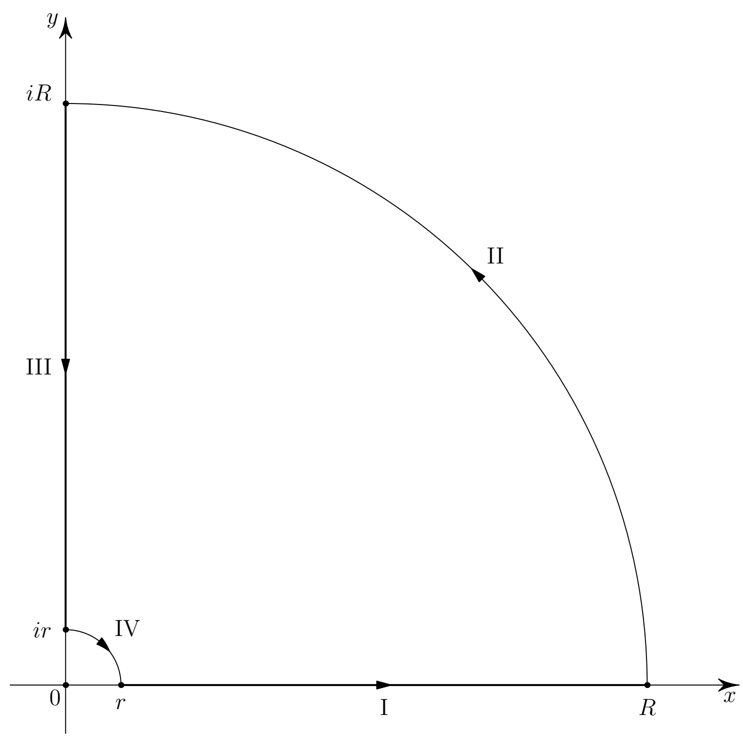

Figure 3: Integration contour for Problem 3

For the absolute value of J_{II} we have the following estimates:

$$\left|J_{\text{II}}\right| = \lim_{R\to\infty}\left|R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{iR\cos\theta}e^{-R\sin\theta}e^{i(1-\alpha)\theta}\,\mathrm{d}\theta\right| \tag{44}$$

$$\leq \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} \left|e^{iR\cos\theta}e^{-R\sin\theta}e^{i(1-\alpha)\theta}\right|\mathrm{d}\theta \tag{45}$$

$$= \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{-R\sin(\theta)}\,\mathrm{d}\theta \leq \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\,\mathrm{d}\theta \tag{46}$$

$$= \lim_{R\to\infty} R^{(1-\alpha)}\frac{\pi}{2R}\int_0^{R} e^{-u}\,\mathrm{d}u = \frac{\pi}{2}\lim_{R\to\infty} R^{-\alpha}\left(1-e^{-R}\right) = 0, \tag{47}$$

where we used the inequalities

$$\sin(\phi) \geq \frac{2}{\pi}\theta \quad\longrightarrow\quad e^{-\sin(\theta)} \leq e^{-\frac{2}{\pi}\theta} \quad\longrightarrow\quad e^{-R\sin(\theta)} \leq e^{-\frac{2R}{\pi}\theta}, \tag{48}$$

that are valid within the integration range $0 \leq \theta \leq \frac{\pi}{2}$, and introduce a new integration variable $u = \frac{2R}{\pi}\theta$.

Thus,

$$J_{\text{II}} = 0. \tag{49}$$

J_{III}: the integration is along the imaginary axis, so $z = iy$, $\mathrm{d}z = i\,\mathrm{d}y$, $r \leq y \leq R$:

$$J_{\text{I}} = \lim_{r\to 0}\lim_{R\to\infty} i^{(1-\alpha)}\int_R^{r}\frac{e^{-y}}{y^{\alpha}}\,\mathrm{d}y = -e^{i\frac{\pi}{2}(1-\alpha)}\int_0^{\infty} e^{-y}\,y^{-\alpha}\,\mathrm{d}y = -e^{i\frac{\pi}{2}(1-\alpha)}\,\Gamma(1-\alpha). \tag{50}$$

J_{IV}: the integration is clockwise along the quarter-circle of radius r, $z = r\,e^{i\theta}$, $\mathrm{d}z = i\,r\,e^{i\theta}\mathrm{d}\theta$, $0 \leq \theta \leq \frac{\pi}{2}$:

$$J_{\text{IV}} = \lim_{r\to 0} i\,r^{(1-\alpha)}\int_{\frac{\pi}{2}}^{0} e^{i\,r\,e^{i\theta}}e^{i(1-\alpha)\theta}\mathrm{d}\theta \approx -\lim_{r\to 0} i\,r^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{i(1-\alpha)\theta}\mathrm{d}\theta = 0. \tag{51}$$

Combining Eqs. (39), (40), and (51), we get

$$J_{\rm I} = e^{i\frac{\pi}{2}(1-\alpha)}\,\Gamma(1-\alpha). \tag{52}$$

Taking the imaginary part, and using Eq. (42), we obtain

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}\,\mathrm{d}x = \sin\left(\frac{\pi}{2}(1-\alpha)\right)\Gamma(1-\alpha). \tag{53}$$

For the case $\alpha = 1$,

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \lim_{\alpha\to 1}\sin\left(\frac{\pi}{2}(1-\alpha)\right)\Gamma(1-\alpha) \approx \lim_{\alpha\to 1}\frac{\pi}{2}(1-\alpha)\Gamma(1-\alpha) = \frac{\pi}{2}\lim_{\alpha\to 1}\Gamma(2-\alpha) = \frac{\pi}{2}\Gamma(1) = \frac{\pi}{2}.$$

4 Fresnel integrals

Problem: Assuming that the value of the Gaussian integral is known,

$$I = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}, \tag{54}$$

evaluate the Fresnel integrals,

$$C = \int_0^\infty \cos\left(x^2\right)\mathrm{d}x \tag{55}$$

and

$$S = \int_0^\infty \sin\left(x^2\right)\mathrm{d}x. \tag{56}$$

The integrals C and S are named after the Fresnel (French physicist, 1788-1827). They were first evaluated by Euler in 1781.

Solution:

Let's pack C and S together:

$$F \equiv C + i\,S = \int_0^\infty \left[\cos\left(x^2\right) + i\cos\left(x^2\right)\right]\mathrm{d}x = \int_0^\infty e^{ix^2}\,\mathrm{d}x, \tag{57}$$

such that

$$C = \operatorname{Re} F \tag{58}$$

and

$$S = \operatorname{Im} F. \tag{59}$$

Consider the integral

$$J = \int_C e^{i z^2}\, \mathrm{d}z, \tag{60}$$

where C is the contour in the complex plane shown in Fig. 4.

Since the integrand in Eq. (60) is analytic inside C,

$$J = 0. \tag{61}$$

On the other hand,

$$J = J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}}, \tag{62}$$

where J_{I} is the integral along the positive real axis, J_{II} is the integral along the circular arc or radius $R \to \infty$, $0 \le \theta \le \frac{\pi}{4}$, and J_{III} is the integral from infinity to the origin along the ray that makes the angle $\theta = \frac{\pi}{4}$ with the real axis.

Let's consider J_{I}, J_{II}, and J_{III} separately:

J_{I}**:** the integration is along the real axis, so $z = x$, $\mathrm{d}z = \mathrm{d}x$, $0 \le x \le \infty$:

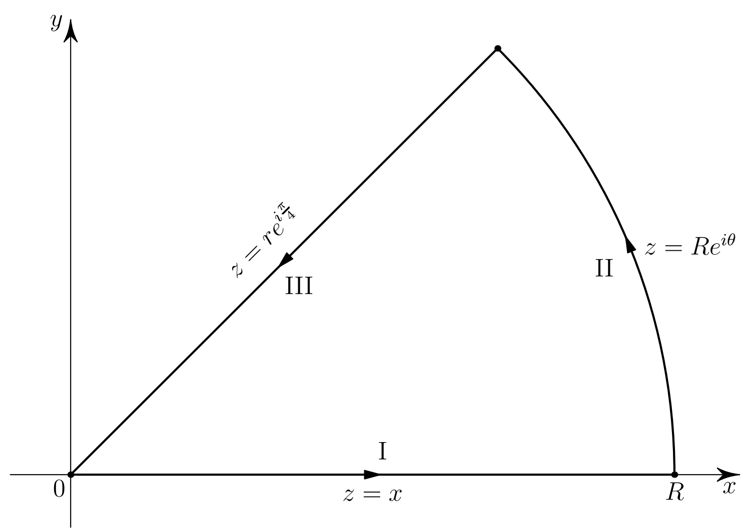

Figure 4: Integration contour for Problem 4

$$J_{\mathrm{I}} = \int\limits_{C_{\mathrm{I}}} e^{i\,z^2}\,\mathrm{d}z = \int\limits_{0}^{\infty} e^{i\,x^2}\,\mathrm{d}x = F. \tag{63}$$

J_{II}: the integration is along the circular arc of radius R so $z = Re^{i\theta}$, $\mathrm{d}z = i\,Re^{i\theta}\mathrm{d}\theta$, $z^2 = R^2\,e^{2i\theta} = R^2\Big(\cos(2\theta) + i\sin(2\theta)\Big)$, $0 \le \theta \le \frac{\pi}{4}$:

$$J_{\mathrm{II}} = \int\limits_{C_{\mathrm{II}}} e^{i\,z^2}\,\mathrm{d}z = i\,R\int\limits_{0}^{\frac{\pi}{4}} e^{i\,R^2\cos(2\theta)}e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta. \tag{64}$$

For the absolute value of J_{II} we have the following estimates:

$$\begin{aligned}
\left|J_{\mathrm{II}}\right| &= \left|R\int\limits_{0}^{\frac{\pi}{4}} e^{i\,R^2\cos(2\theta)}e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta\right| \le R\int\limits_{0}^{\frac{\pi}{4}}\left|e^{i\,R^2\cos(2\theta)}e^{-R^2\sin(2\theta)}\right|\mathrm{d}\theta & (65)\\
&= R\int\limits_{0}^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta = \frac{R}{2}\int\limits_{0}^{\frac{\pi}{2}} e^{-R^2\sin(\phi)}\,\mathrm{d}\phi < \frac{R}{2}\int\limits_{0}^{\frac{\pi}{2}} e^{-\frac{2R^2}{\pi}\phi}\,\mathrm{d}\phi & (66)\\
&= \frac{R}{2}\,\frac{\pi}{2R^2}\int\limits_{0}^{R^2} e^{-u}\,\mathrm{d}u = \frac{\pi}{4R}\Big(1 - e^{-R^2}\Big) < \frac{\pi}{4R}, & (67)
\end{aligned}$$

where we introduced a new integration variable $\phi = 2\theta$, used the inequalities

$$\sin(\phi) \ge \frac{2}{\pi}\phi \quad\longrightarrow\quad e^{-\sin(\phi)} \le e^{-\frac{2}{\pi}\phi} \quad\longrightarrow\quad e^{-R^2\sin(\phi)} \le e^{-\frac{2R^2}{\pi}\phi}, \tag{68}$$

that are valid within the integration range $0 \le \phi \le \frac{\pi}{2}$, and introduce a new integration variable $u = \frac{2R^2}{\pi}\phi$.

Thus we obtained that

$$\left|J_{\mathrm{II}}\right| < \frac{\pi}{4R}. \tag{69}$$

Therefore,

$$J_{\mathrm{II}} = 0 \tag{70}$$

as $R \to \infty$.

J_{III}: the integration is along the ray making the angle $\frac{\pi}{4}$ with the real axis so $z = re^{i\frac{\pi}{4}}$, $z^2 = r^2 e^{i\frac{\pi}{2}} = i\,r^2$, $\mathrm{d}z = e^{i\frac{\pi}{4}}\,\mathrm{d}r$, $0 \leq r < \infty$.

$$J_{\text{III}} = \int\limits_{C_{\text{III}}} e^{i\,z^2}\,\mathrm{d}z = e^{i\frac{\pi}{4}} \int\limits_{\infty}^{0} e^{-r^2}\,\mathrm{d}r = -e^{i\frac{\pi}{4}} \int\limits_{0}^{\infty} e^{-r^2}\,\mathrm{d}r = -e^{i\frac{\pi}{4}}\,\frac{\sqrt{\pi}}{2}. \tag{71}$$

Combining Eqs. (61), (63), (70), and (71) we obtain:

$$F = \frac{\sqrt{\pi}}{2}\,e^{-\frac{\pi}{4}}. \tag{72}$$

Finally, the Fresnel's integrals are:

$$C = \operatorname{Re} F = \frac{\sqrt{\pi}}{2}\cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{8}} \tag{73}$$

and

$$S = -\operatorname{Im} F = \frac{\sqrt{\pi}}{2}\sin\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{8}}. \tag{74}$$