

# LAPLACE'S METHOD FOR INTEGRALS

FALL SEMESTER 2020

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Laplace's method is a general technique for obtaining the asymptotic behavior of integrals in which the large parameter  $\lambda$ ,  $\lambda \rightarrow \infty$ , appears in the exponent:

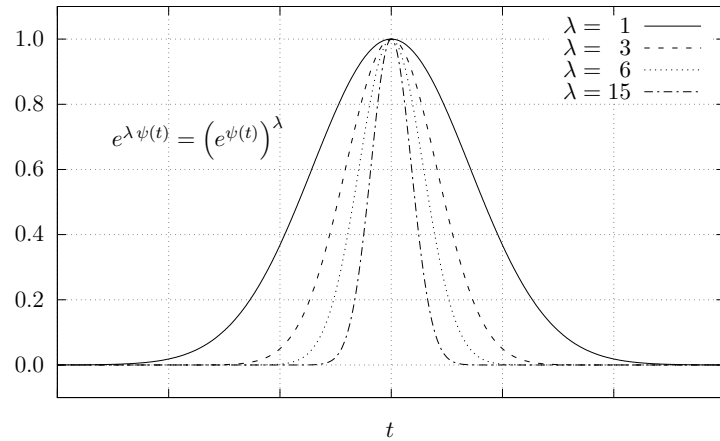
$$I(\lambda) = \int_a^b f(t) e^{\lambda \phi(t)} dt = \int_a^b f(t) \left( e^{\phi(t)} \right)^\lambda dt. \quad (1)$$

Here  $f(t)$  and  $\phi(t)$  are real continuous functions, independent of  $\lambda$ . Integrals of this form are called *Laplace integrals*. Laplace's method relies on the following observation: if the real continuous function  $\phi(t)$  has its maximum on the interval  $a \leq t \leq b$  at  $t = t_0$  and if  $f(t_0) \neq 0$ , then it is only the immediate neighborhood of  $t = t_0$  that contributes to the asymptotic expansion of  $I(\lambda)$  for large  $\lambda$ .

Indeed, we can always write  $e^{\phi(t)}$  as  $e^{\phi(t_0)} e^{\psi(t)}$ , where  $e^{\phi(t_0)}$  is just a constant multiplication factor that can be factored out of the integral. Here we defined  $\psi(t) \equiv \phi(t) - \phi(t_0)$ . The maximal value of  $\psi(t)$  is zero, thus the maximal value of  $e^{\psi(t)}$  is one. A typical behavior of  $e^{\psi(t)}$  is sketched in Fig. 1 in solid line. As we raise  $e^{\psi(t)}$  into power  $\lambda$ , its maximum stays “fixed” at  $(x_0, 1)$  but its wings are “moving down” toward the  $x$  axis, thus making the graph narrower (see Fig. 1). Therefore we can replace  $f(t)$  and  $\phi(t)$  with their approximations that need to be good ones only in the vicinity of  $t_0$ .

The logic of the Laplace method works without changes for a more general form of the

Figure 1: Changes of the integrand in Laplace integral as the parameter  $\lambda$  is increasing.



integrand:

$$\int_a^b f(t) (\kappa(t))^\lambda dt, \quad (2)$$

where  $\lambda$  is a large parameter as before,  $\lambda \rightarrow \infty$ ,  $\kappa(t)$  is real continuous function that is independent of  $\lambda$ , doesn't change the sign on the interval of integration  $a \leq t \leq b$  (say it is positive), and has a single maximum on that interval.

Let's consider simple examples of Laplace's method.

**Example 1.** Find the leading term of the asymptotics of the following integral for  $\lambda \rightarrow \infty$ :

$$I(\lambda) = \int_0^{\frac{\pi}{2}} \frac{e^{\lambda \cos(x)}}{x^2 + 4} dx. \quad (3)$$

Since only small  $|x|$ , such that  $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ , are important in the integral Eq. (11), we can approximate the integrand as following:

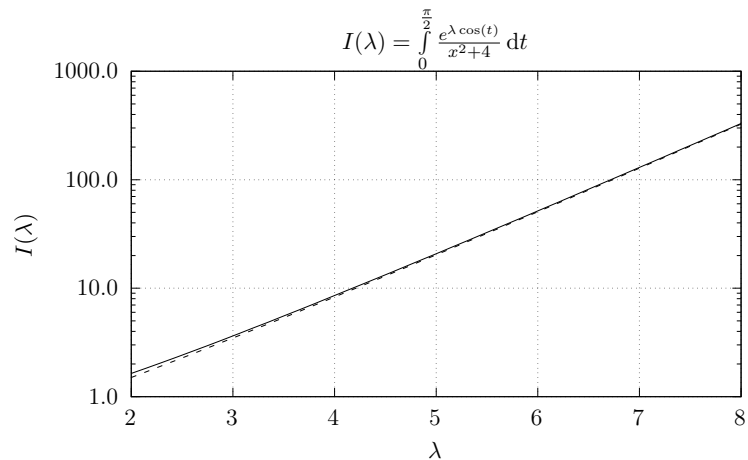
$$\cos(x) \approx 1 - \frac{x^2}{2}, \quad \longrightarrow \quad e^{\lambda \cos(x)} \approx e^\lambda e^{-\frac{\lambda}{2}x^2}, \quad (4)$$

$$\frac{1}{x^2 + 4} \approx \frac{1}{4}. \quad (5)$$

Thus,

$$I(\lambda) \sim \frac{1}{4} e^\lambda \int_0^\infty e^{-\frac{\lambda}{2}x^2} dx = \frac{1}{8} \sqrt{\frac{2\pi}{\lambda}} e^\lambda = \boxed{\sqrt{\frac{\pi}{32\lambda}} e^\lambda}. \quad (6)$$

Figure 2: Asymptotics Eq. (6) (solid line) compared to the numerically evaluated integral (3) (dashed line) for  $2 \leq x \leq 8$ . Notice the logarithmic scale on  $y$  axis.



**Example 2.** Find the leading term of the asymptotics of the following integral for  $\lambda \rightarrow \infty$ :

$$I(\lambda) = \int_0^1 e^{-\lambda \sin^3(x)} dx. \quad (7)$$

The maximum of the function in the exponent,  $e^{-\sin^3 x}$  is at  $x = 0$ , so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range,  $x = 0$ , where  $\sin^3 x \sim x^3$ .

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x^3} dx. \quad (8)$$

To evaluate the last integral, let's introduce a new integration variable,  $u = \lambda x^3$ :

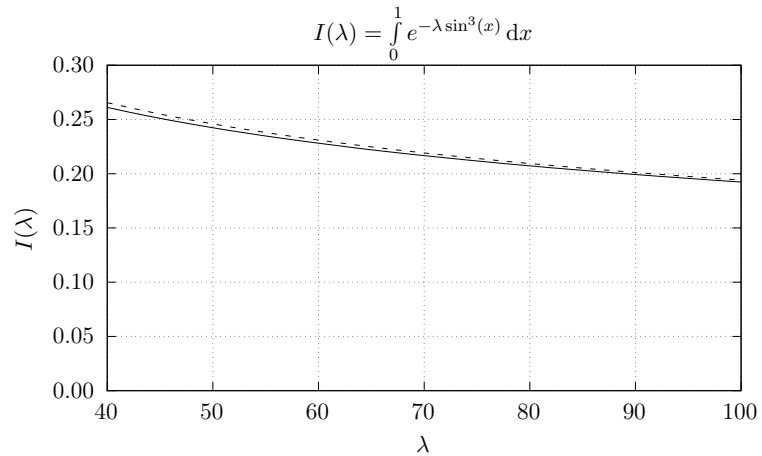
$$x^3 = \frac{u}{\lambda} \longrightarrow x = \frac{u^{\frac{1}{3}}}{\lambda^{\frac{1}{3}}} \longrightarrow dx = \frac{u^{\frac{1}{3}-1}}{3\lambda^{\frac{1}{3}}} du. \quad (9)$$

$$I(\lambda) \sim \frac{1}{3\lambda^{\frac{1}{3}}} \int_0^\infty e^{-u} u^{\frac{1}{3}-1} du = \frac{\Gamma(\frac{1}{3})}{3\lambda^{\frac{1}{3}}} = \boxed{\frac{\Gamma(\frac{4}{3})}{\lambda^{\frac{1}{3}}}}. \quad (10)$$

**Example 3.** Find the leading term of the asymptotics of the following integral for  $\lambda \rightarrow \infty$ :

$$I(\lambda) = \int_{-3}^4 e^{-\lambda x^2} \log(1+x^2) dx. \quad (11)$$

Figure 3: Asymptotics Eq. (10) (solid line) compared to the numerically evaluated integral (7) (dashed line) for  $40 \leq \lambda \leq 100$ .



Since only small  $|x|$ , such that  $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ , are important in the integral Eq. (11) (for  $|x| \geq \frac{1}{\sqrt{\lambda}}$  the integrand is negligibly small due to the exponent's factor), we can approximate the function in the integrand as following:

$$\log(1 + x^2) \sim x^2. \quad (12)$$

Thus,

$$I(\lambda) \sim \int_{-3}^4 e^{-\lambda x^2} x^2 dx \sim \int_{-\infty}^{\infty} e^{-\lambda x^2} x^2 dx = 2 \int_0^{\infty} e^{-\lambda x^2} x^2 dx. \quad (13)$$

Introducing the new integration variable  $u$ ,

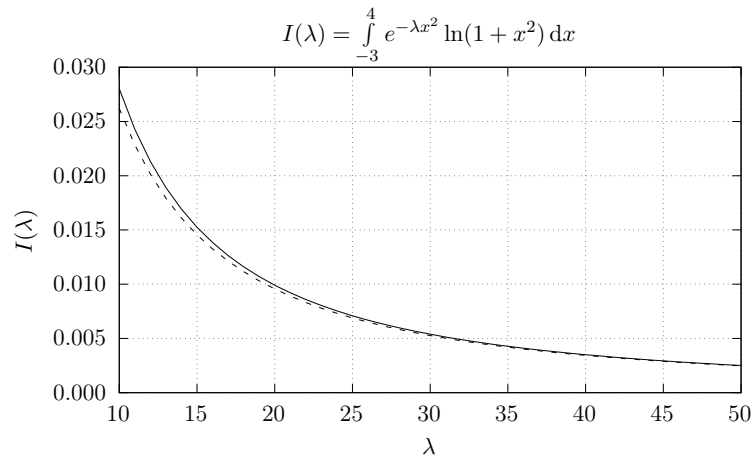
$$u = \lambda x^2 \quad \longrightarrow \quad x^2 = \frac{u}{\lambda} \quad \longrightarrow \quad x = \frac{1}{\sqrt{\lambda}} u^{\frac{1}{2}} \quad \longrightarrow \quad dx = \frac{1}{2\sqrt{\lambda}} u^{-\frac{1}{2}} du. \quad (14)$$

$$I(\lambda) \sim \lambda^{-\frac{3}{2}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}} du = \lambda^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) = \lambda^{-\frac{3}{2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{2} \lambda^{-\frac{3}{2}}} \quad (15)$$

**Example 4.** Find the leading term of the asymptotics of the following integral for  $n \gg 1$ :

$$I(n) = \int_{-1}^1 (\cos x)^n dx, \quad (16)$$

Figure 4: Asymptotics Eq. (15) (solid line) compared to the numerically evaluated integral Eq. (11) (dashed line) for  $10 \leq \lambda \leq 50$ .



Since only small  $|x|$ , such that  $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ , are important in the integral, we can approximate

$$\cos x \sim 1 - \frac{x^2}{2} \sim e^{-\frac{x^2}{2}}. \quad (17)$$

Thus,

$$I(n) = \int_{-1}^1 \left( e^{-\frac{x^2}{2}} \right)^n dx \sim \int_{-\infty}^{\infty} e^{-\frac{nx^2}{2}} dx = \int_{-\infty}^{\infty} e^{-(\sqrt{\frac{n}{2}}x)^2} dx = \quad (18)$$

$$= \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-(\sqrt{\frac{n}{2}}x)^2} d\left(\sqrt{\frac{n}{2}}x\right) = \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-u^2} du = \quad (19)$$

$$= \boxed{\sqrt{\frac{2\pi}{n}}}. \quad (20)$$

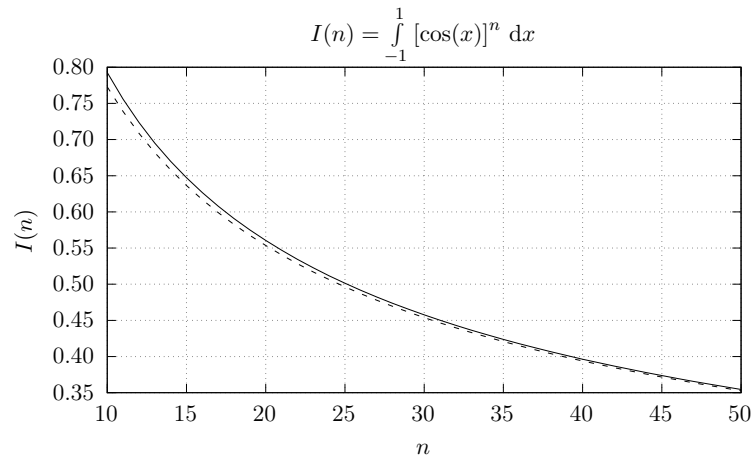
**Example 5.** Find the leading term of the asymptotics of gamma function,  $\Gamma(x)$ , for  $x \rightarrow \infty$ :

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt = \int_0^{\infty} e^{-t+x \log t} \frac{1}{t} dt \quad (21)$$

The function in the exponent in Eq. (21),

$$f(t) = -t + x \log t, \quad (22)$$

Figure 5: Asymptotics Eq. (20) (solid line) compared to the numerically evaluated integral (16) (dashed line) for  $10 \leq n \leq 50$ .



has its maximum at  $t = t_0$  which depends upon  $x$ :

$$\frac{df}{dt} = 0, \quad \longrightarrow \quad -1 + \frac{x}{t} = 0 \quad \longrightarrow \quad t_0 = x. \quad (23)$$

To make the maximum independent of  $x$ , let's introduce a new integration variable,  $s$ ,

$$s = \frac{t}{x}, \quad \longrightarrow \quad t = xs, \quad \longrightarrow \quad dt = x ds, \quad \frac{dt}{t} = \frac{ds}{s}, \quad (24)$$

$$f(t) = -t + x \log t = -xs + x \log s + x \log x. \quad (25)$$

$$\Gamma(x) = e^{x \log x} \int_0^\infty e^{-x(s - \log s)} \frac{1}{s} ds. \quad (26)$$

Let's apply the Laplace's method to the integral in Eq. (26):

$$f(s) = s - \log s, \quad \frac{df}{ds} = 1 - \frac{1}{s}. \quad (27)$$

$$\frac{df}{ds} = 0, \quad \longrightarrow \quad s_0 = 1. \quad (28)$$

$$f(s_0) = 1, \quad \frac{d^2 f}{ds^2} = \frac{1}{s^2}, \quad \longrightarrow \quad \frac{d^2 f}{ds^2}(s_0) = 1. \quad (29)$$

$$f(s) \approx f(s_0) + \frac{1}{2} \frac{d^2 f}{ds^2}(s_0)(s - s_0)^2 = 1 + \frac{1}{2}(s - 1)^2. \quad (30)$$

$$\int_0^\infty e^{-x f(s)} \frac{1}{s} ds \sim \int_0^\infty e^{-x(1 + \frac{1}{2}(s-1)^2)} \frac{1}{s_0} ds \sim e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}x(s-1)^2} ds = e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}xs^2} ds. \quad (31)$$

The last integral is a Gaussian one:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}xs^2} ds = \sqrt{\frac{2\pi}{x}}, \quad (32)$$

therefore

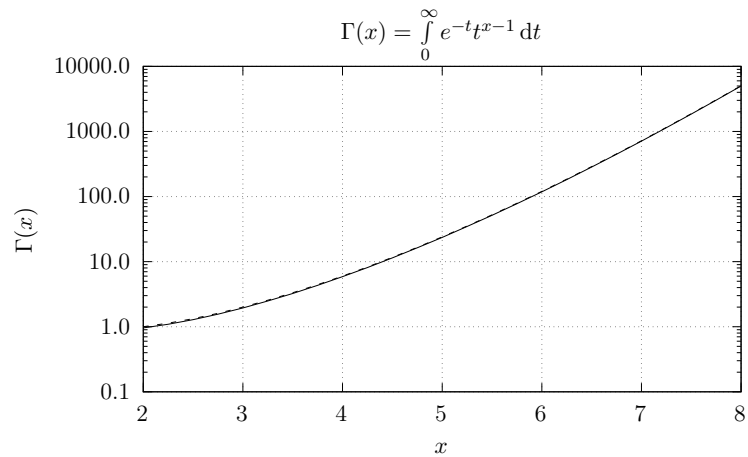
$$\int_0^{\infty} e^{-x(s-\log s)} \frac{1}{s} ds \sim e^{-x} \sqrt{\frac{2\pi}{x}}. \quad (33)$$

Finally, combining Eq. (26) and Eq. (33)

$$\Gamma(x) \sim e^{x \log x} e^{-x} \sqrt{\frac{2\pi}{x}} = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x. \quad (34)$$

The expression Eq. (34) is the leading term of the *Stirling approximation* for Gamma function.

Figure 6: Asymptotics Eq. (34) (solid line) compared to the numerically evaluated integral (21) (dashed line) for  $2 \leq x \leq 8$ . Notice the logarithmic scale on  $y$  axis.



## References

- [1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.