KELVIN'S SHIP WAVE PATTERN

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When a disturbance (e.g. a ship) travels on a water surface, it carries with it a familiar pattern of bow and stern waves which was first explained mathematically by Lord Kelvin in 1891. He considered the waves generated by a prescribed pressure distribution moving with a constant velocity U and acting on the water surface. The magnitude of the pressure and the resulting wave slope were assumed to be small, so that the equations of motion could be linearized; the viscosity of water and the surface tension were neglected. Under these assumptions it is sufficient (in principle) to calculate the waves due to a moving concentrated pressure point.

In the section on *Integrals with Coalescing Saddles*, the Digital Library of Mathematical Functions (DLMF) provides the analysis of the Kelvin wake pattern, starting from the following equations derived in A Ship's Wake, by D. Longcope.

The change due to waves in the water height, $z(\mathbf{x}, t)$, at a point $\mathbf{x}$ at time t can be written in the most general way as the Fourier integral

$$z(\mathbf{r}, t) = \int A(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, \mathrm{d}^2\mathbf{k} \tag{1}$$

for some unspecified yet amplitude $A(\mathbf{k})$. Here $\omega(\mathbf{k}$ is the frequency of *gravity waves* in the water reference frame.

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|}. \tag{2}$$

In the ship's reference frame the waves' frequency is

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|} - \mathbf{u}\cdot\mathbf{k}. \tag{3}$$

where $\mathbf{k}$ is the wave vector, g is the acceleration of gravity, and $\mathbf{u}$ is the ship velocity. Asking for stationary waves in the ship frame with $\omega(\mathbf{k}) = 0$, we reduce the wave vector integration in Eq. (1) to one dimensional integration (i.e. by an angle) by using the relation

$$\mathbf{k} = \frac{g}{(\mathbf{u}\cdot\hat{\mathbf{k}})^2}\,\hat{\mathbf{k}}, \tag{4}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$,

$$\hat{\mathbf{k}} = (\cos\theta, \sin\theta). \tag{5}$$

The angle θ is measured with respect to the direction of $\mathbf{u}$:

$$\mathbf{u} \cdot \hat{\mathbf{k}} = u \cos\theta. \tag{6}$$

The height is then given by

$$z(\mathbf{r}) = \int_0^{2\pi} A(\theta) \exp\left[i \frac{g}{(\mathbf{u} \cdot \hat{\mathbf{k}})^2} \hat{\mathbf{k}} \cdot \mathbf{r} \right] \mathrm{d}\theta. \tag{7}$$

The information about the ship is encoded in the amplitudes $A(\theta_k)$. DLMF assumes a point ship radiating backwards: A is constant over $\theta \in (-\pi/2, \pi/2)$, and zero outside it. Rewriting the integral in polar coordinates, r and ϕ,

$$\mathbf{r} = r(\cos\phi, -\sin\phi), \tag{8}$$

and introducing the dimensionless distance $\rho = \frac{g}{u^2} r$, we obtain

$$z(\phi, \rho) = \int_{-\pi/2}^{\pi/2} \cos\left(\rho \frac{\cos(\theta + \phi)}{\cos^2\theta} \right) \mathrm{d}\theta. \tag{9}$$

The wake pattern calculated using the integral Eq. (9) is presented in Fig. (1).

General references

[1] Fritz Ursell. "On Kelvin's ship-wave pattern". In: *Journal of Fluid Mechanics* 8.3 (1960), pp. 418–431.

[2] *NIST Digital Library of Mathematical Functions, §36.13 Kelvin's Ship-Wave Pattern*. F. W. J. Olver, et al. eds. URL: https://dlmf.nist.gov/36.13.

Student projects' reports

[3] Ksenija Maver. *Kelvin ship waves*. Department of Physics, University of Ljubljana. 2004. URL: http://www-f1.ijs.si/~rudi/sola/Kelvin_wave%201.pdf.

[4] Špela Rožman. *Wake pattern of a boat*. Department of Physics, University of Ljubljana. 2009. URL: https://www.prirodopolis.hr/daily_phy/pdf/speed.pdf.

[5] Joy Perkinson. *Gravity waves: the Kelvin wedge and related problems*. 2010. URL: http://web.mit.edu/joyp/Public/HarvardWork/AM201FinalPaper.pdf.

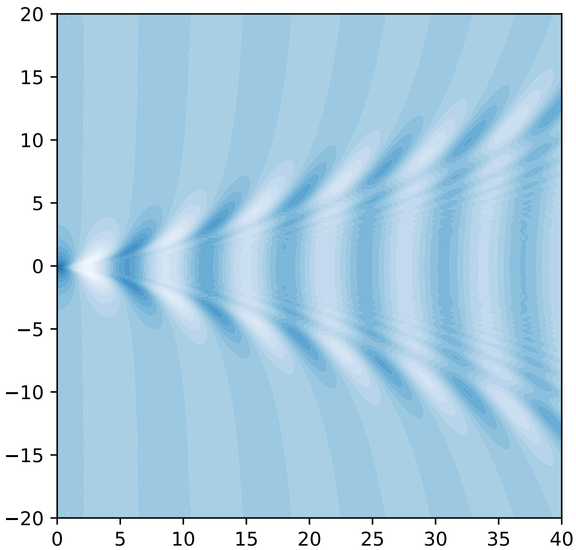

Figure 1: Kelvin wake calculated using Eq. (9)

Talks and lecture notes

[6] Karima Khusnutdinova. *Kelvin's method of stationary phase and the ship wave pattern*. 2010. URL: http://web.archive.org/web/20150515070140/http://homepages.lboro.ac.uk/~makk/MathRev_Kelvin.pdf.

[7] Emilio Pisanty. *Ship wake height*. Physics Stack Exchange. 2013. URL: https://physics.stackexchange.com/q/76496.

[8] Andrew French. *The shape of ship wakes: the Kelvin wedge*. 2013. URL: http://www.eclecticon.info/index_htm_files/Kelvin%20wedge.ppt.

[9] Dana Longcope. *A ship's wake*. Department of Physics, Montana State University. 2015. URL: http://solar.physics.montana.edu/dana/ship_wake.pdf.