Name: ______________________________

Date: ______________________________

Collaborators: ____

Question:	1	2	3	4	Total
Points:	15	25	20	20	80
Score:					

Instructor/grader comments:

Cauchy-Riemann equations

1. (15 points) Use Cauchy-Riemann equations to find the analytic function $f(z)$, $z = x+iy$, such that its real part is as following:

$$\operatorname{Re} f(z) = u(x,y) = e^x \sin y,$$

and

$$f(i\pi) = 0.$$

Express the result for $f(z)$ as a **function of z only.**

Answer: $f(z) = -i\,(e^z + 1)$.

The Cauchy integral theorem

2. (25 points) Evaluate the integral

$$I = \int_0^\infty \sin\left(x^3\right) \mathrm{d}x$$

Hints: consider the integral

$$\oint_C e^{-z^3}\,\mathrm{d}z$$

along the contour C sketched in Fig. 1; use the Euler formula; use the fact that

$$\int_0^\infty e^{-x^3}\,d\mathrm{x} \equiv \Gamma\left(\frac{4}{3}\right),$$

where Γ is gamma function. (Can you show this?)

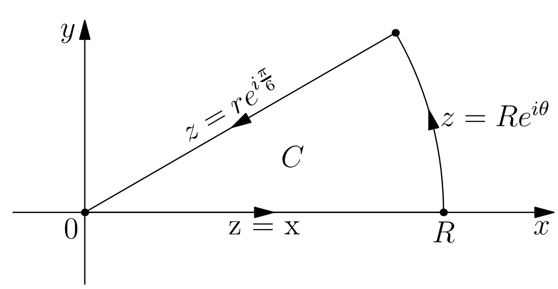

Figure 1: Integration contour for Problem 2. ($R \to \infty$) .

Answer: $I = \frac{1}{2}\Gamma\left(\frac{4}{3}\right)$.

Method of residues

3. (20 points) Calculate the integral for real a, $|a| < 1$, and integer non-negative n:

$$I = \int_{-\pi}^{\pi} \frac{\cos(n\varphi)}{1 - 2a\cos(\varphi) + a^2}\, \mathrm{d}\varphi.$$

Sketch the integration contour. Indicate the position(s) of the pole(s) of the integrand. Compare your answer with the result produced by a computer algebra system.

Integral to stump a computer algebra system

4. (a) (15 points) Construct an definite integral that you can evaluate analytically but a computer algebra can not. Use the method described in the handout "The integral that stumped Feynman". Do not use the integrands similar to ones discussed in the handout. Use a computer algebra system (CAS) for finding the real and the imaginary parts of your complex expressions.

   (b) (5 points) To verify your result, numerically evaluate your integral and your answer. Use a computer algebra system (CAS) for numerics.

   Enclose a printout of you CAS session.