Name: _____

Date: _____

Collaborators: _____

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	30	5	5	100
Score:							

Instructor/grader comments:

Euler's formula

1. (20 points) Evaluate the following sum:

$$C(\theta) = \cos(\theta) + \cos(2\theta) + \ldots + \cos(n\theta).$$

Answer:

$$C(x) = \frac{1}{2}\left[\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} - 1\right]$$

Gamma and Beta functions

2. Evaluate the following expressions. Here Γ and B are Euler gamma and beta functions. Only the values of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$ are known.

   (a) (4 points) $\Gamma\left(\frac{5}{2}\right)$

   (b) (8 points) $\Gamma\left(-\frac{3}{2}\right)$

   (c) (4 points) $\Gamma(5)$

   (d) (4 points) $B\left(\frac{1}{2}, \frac{5}{2}\right)$

Leibniz' rule

3. (20 points) Find the positive value of x that maximizes the value of the following integral

$$I(x) = \int_{x-1}^{x+1} \frac{\mathrm{d}u}{\Gamma(u)}.$$

Hints: Find the derivative of $I(x)$. What is its value at the maximum of $I(x)$? Simplify the equation that you obtained using the relation $\Gamma(x+1) = x\Gamma(x)$. Solve the equation and select the correct solution.

Answer: $x = \frac{1+\sqrt{5}}{2}$

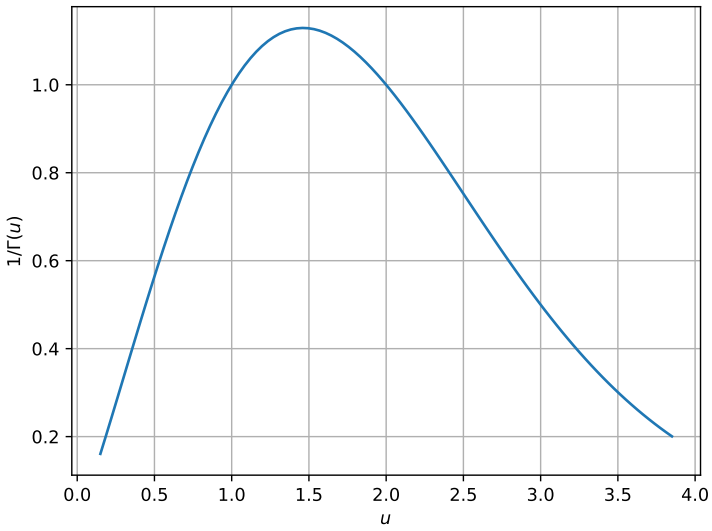

Figure 1: Graph of $1/\Gamma(u)$.

Differentiation with respect to a parameter

4. (30 points) Evaluate the following integral

$$I = \int_0^\infty e^{-\left(x^2+\frac{1}{x^2}\right)}\,\mathrm{d}x$$

Hints: consider the integral $I(\alpha) = \int_0^\infty e^{-\left(x^2+\frac{\alpha}{x^2}\right)}\,\mathrm{d}x$. What you need is $I(1)$ and what you know is $I(0)$ $\left(I(0) = \int_0^\infty e^{-x^2}\,\mathrm{d}x\right)$. Take the derivative of $I(\alpha)$ and transform the integral by introducing the new integration variable $u = \frac{\sqrt{\alpha}}{x}$. Solve the differential equation you thus obtained.

Answer: $I = \frac{\sqrt{\pi}}{2}e^{-2}$

Computer algebra

5. (5 points) Use a computer algebra system to expand the following function, $f(x)$, into a power series about $x = 0$. Keep the terms up to $\sim x^3$.

$$f(x) = e^{\sqrt{\sin(x)}}$$

Print your mathematica session and attach the printout to the rest of your homework.

6. (5 points) Use a computer algebra system to evaluate the following indefinite integral:

$$\int \frac{x}{1+\sin x}\,\mathrm{d}x$$

Print your mathematica session and attach the printout to the rest of your homework.