EVALUATE GAUSSIAN AND FRESNEL'S INTEGRALS USING DIFFERENTIATION UNDER THE INTEGRAL SIGN

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In this note we evaluate Gaussian integral,

$$I_g \equiv \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}, \tag{1}$$

and Fresnel integrals,

$$I_c \equiv \int_0^\infty \cos\left(x^2\right)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}} \tag{2}$$

and

$$I_s \equiv \int_0^\infty \sin\left(x^2\right)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}, \tag{3}$$

using differentiation under the integral sign.

The two parts of this note can be read independently.

Gaussian integral

Consider the following integral

$$J(x) = \int_0^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2}\,\mathrm{d}y. \tag{4}$$

We know its values for $x = 0$:

$$J(0) = \int_0^\infty \frac{\mathrm{d}y}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \tag{5}$$

and for $x = \infty$:

$$J(\infty) = 0. \tag{6}$$

The derivative of $J(x)$,

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}x} &= -2\,x \int_0^\infty e^{-x^2(1+y^2)}\,\mathrm{d}y = -2\,e^{-x^2} \int_0^\infty e^{-(xy)^2}\,\mathrm{d}(xy) \\ &= -2\,e^{-x^2} \int_0^\infty e^{-u^2}\,\mathrm{d}u \\ &= -2\,e^{-x^2} I_g. \end{aligned} \tag{7}$$

Integrating Eq. (7) with respect to x from 0 to ∞, we obtain:

$$J(\infty) - J(0) = -2I_g \int_0^\infty e^{-x^2}\,\mathrm{d}x = -2I_g^2, \tag{8}$$

or, using Eq. (5) and Eq. (6),

$$I_g^2 = \frac{J(0)}{2} = \frac{\pi}{4}, \tag{9}$$

i.e.

$$I_g = \frac{\sqrt{\pi}}{2}. \tag{10}$$

Fresnel's integrals

Consider the following integrals:

$$c(x) = \int_0^\infty \frac{\cos\left(x^2(1+y^2)\right)}{1+y^2}\,\mathrm{d}y, \tag{11}$$

$$s(x) = \int_0^\infty \frac{\sin\left(x^2(1+y^2)\right)}{1+y^2}\,\mathrm{d}y. \tag{12}$$

We know their values for $x = 0$:

$$c(0) = \int_0^\infty \frac{\mathrm{d}y}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \tag{13}$$

$$s(0) = 0, \tag{14}$$

and for $x = \infty$:

$$c(\infty) = s(\infty) = 0. \tag{15}$$

The derivative of $c(x)$,

$$\begin{aligned}
\frac{\mathrm{d}c}{\mathrm{d}x} &= -2\,x \int_0^\infty \sin\left(x^2(1+y^2)\right)\mathrm{d}y = -2\,x \int_0^\infty \sin\left(x^2 + (xy)^2\right)\mathrm{d}y \\
&= -2\,\sin(x^2) \int_0^\infty \cos(xy)^2\,\mathrm{d}(xy) - 2\,\cos(x^2) \int_0^\infty \sin(xy)^2\,\mathrm{d}(xy) \\
&= -2\,\sin(x^2) \int_0^\infty \cos(u)^2\,\mathrm{d}u - 2\,\cos(x^2) \int_0^\infty \sin(u^2)\,\mathrm{d}u \\
&= -2\,\sin(x^2)\,I_c - 2\,\cos(x^2)\,I_s.
\end{aligned} \tag{16}$$

Similarly,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = 2\,\cos(x^2)\,I_c - 2\,\sin(x^2)\,I_s. \tag{17}$$

Integrating Eqs. (16), (17) with respect to x from 0 to ∞, we obtain:

$$c(\infty) - c(0) = -2\, I_c \int_0^\infty \sin(x^2)\,\mathrm{d}x - 2\, I_s \int_0^\infty \cos(x^2)\,\mathrm{d}x = -4 I_c\, I_s, \tag{18}$$

$$s(\infty) - s(0) = 2\, I_c \int_0^\infty \cos(x^2)\,\mathrm{d}x - 2\, I_s \int_0^\infty \sin(x^2)\,\mathrm{d}x = 2\, I_c^2 - 2\, I_s^2. \tag{19}$$

Using the boundary conditions Eqs. (13)-(15), $c(\infty) - c(0) = -\frac{\pi}{2}$, $s(\infty) - s(0) = 0$, therefore

$$I_c\, I_s = \frac{\pi}{8}, \quad I_c = I_s. \tag{20}$$

Finally,

$$I_c = I_s = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{21}$$

References

[1] J. van Yzeren, "Moivre's and Fresnel's integrals by simple integration," *The American Mathematical Monthly*, vol. 86, no. 8, pp. 690–693, 1979.