EULER'S FORMULA

FALL SEMESTER 2020

https://www.phys.uconn.edu/~rozman/Courses/P2400_20F/

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1 Introduction

In complex analysis Euler's formula establishes the relation between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number θ:

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1}$$

where e is the base of the natural logarithm, i is the imaginary unit, $i = \sqrt{-1}$, and $\cos\theta$ and $\sin\theta$ are the trigonometric functions, with the argument θ given in radians. The formula Eq. (1) was published by Euler in 1748, obtained by comparing the series expansions of the exponential and trigonometric expressions.

2 Derivation

Consider

$$z(\theta) = \cos\theta + i\sin\theta, \tag{2}$$

where θ is a real parameter.

Let's take the derivative of z with respect to θ:

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\sin\theta + i\cos\theta = i\left(\cos\theta + i\sin\theta\right) = iz. \tag{3}$$

Eq. (3) is a first order ordinary differential equation. Separating the variables and integrating, we obtain:

$$\frac{\mathrm{d}z}{z} = i\,\mathrm{d}\theta, \tag{4}$$

$$\ln z = i\theta + C', \tag{5}$$

where C' is an integration constant.

Exponentiating Eq. (5), we get:

$$z(\theta) = Ce^{i\theta}, \tag{6}$$

where $C = e^{C'}$ is just another constant.

From Eq. (2),

$$z(0) = 1. \tag{7}$$

From Eq. (6)

$$z(0) = C. \tag{8}$$

Therefore, $C = 1$ and

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9}$$

For $\theta = \pi$ Eq. (9) establishes the relation between the three fundamental constants — e, π, and i:

$$\boxed{e^{i\pi} = -1}. \tag{10}$$

3 Geometric interpretation and the complex plane

A point in the complex plane can be represented by a complex number written in cartesian coordinates. Euler's formula provides a means of conversion between cartesian coordinates and polar coordinates. The polar form simplifies the mathematics when used in multiplication, division, or powers of complex numbers.

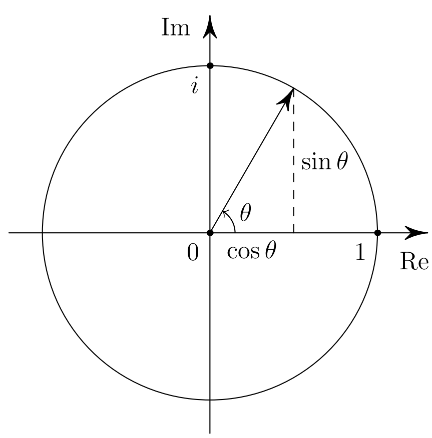

Figure 1: Euler's formula illustrated in the complex plane.