DIFFERENTIATING UNDER THE INTEGRAL SIGN

Fall semester 2020

https://www.phys.uconn.edu/~rozman/Courses/P2400_20F/

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1 The case of the integrand depending on a parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b f(x,t)\,\mathrm{d}x\right) = \int_a^b \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x \tag{1}$$

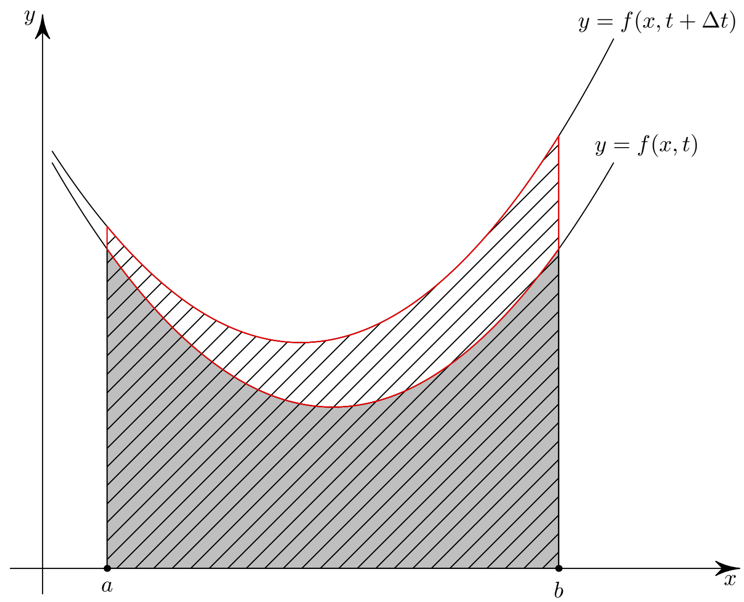
$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x \tag{2}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t\to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} \tag{3}$$

$$I(t+\Delta t) - I(t) = \int_a^b \Big[f(x,t+\Delta t) - f(x,t)\Big]\,\mathrm{d}x \tag{4}$$

$$= \int_a^b \left[\frac{\partial f}{\partial t}\Delta t + O\left(\Delta t^2\right)\right]\mathrm{d}x = \left[\int_a^b \left(\frac{\partial f}{\partial t}\right)\mathrm{d}x\right]\Delta t + O\left(\Delta t^2\right) \tag{5}$$

Figure 1: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (4).



2 Case of the integration range depending on a parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_{a(t)}^{b(t)} f(x)\,\mathrm{d}x\right) = f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{6}$$

$$I(t) = \int\limits_{a(t)}^{b(t)} f(x)\,\mathrm{d}x \tag{7}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} \tag{8}$$

$$\begin{aligned} I(t+\Delta t) - I(t) &= \left(b(t+\Delta t) - b(t)\right) f\left(b(t)\right) - \left(a(t+\Delta t) - a(t)\right) f\left(a(t)\right) + O\left(\Delta t^2\right) & (9) \\ &= \left[f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t}\right]\Delta t + O\left(\Delta t^2\right) & (10) \end{aligned}$$

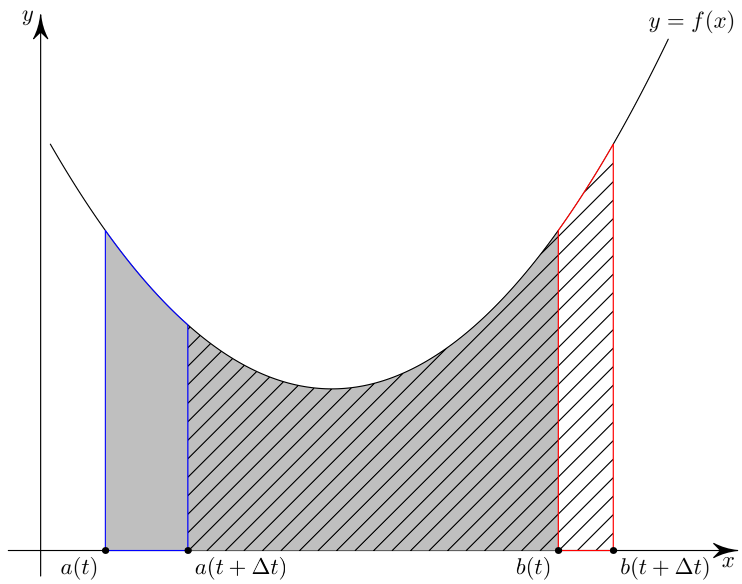


Figure 2: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (9).

3 General case

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x + f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{11}$$

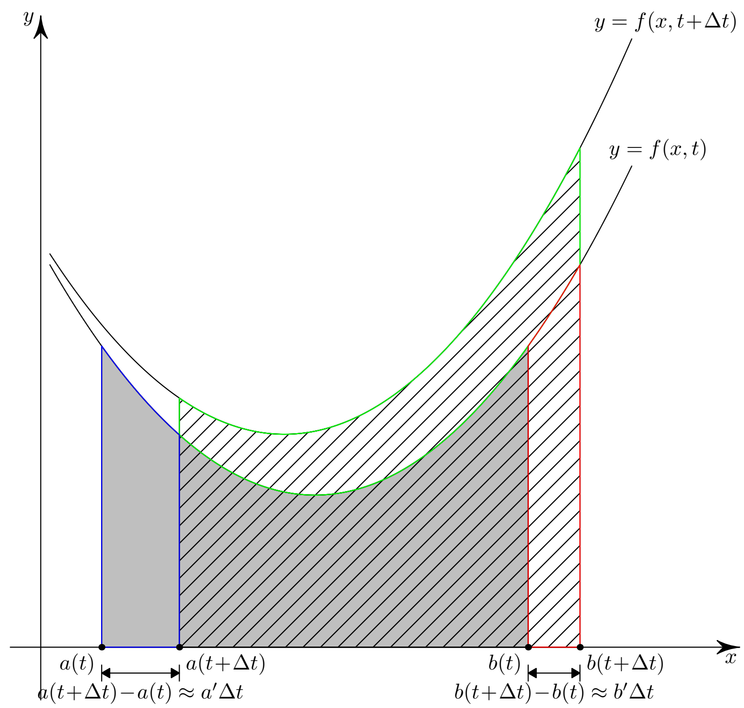


Figure 3: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (9).