INDUCED EMF IN A CIRCULAR LOOP

Fall semester 2020

https://www.phys.uconn.edu/~rozman/Courses/P2400_20F/

Last modified: October 4, 2020

A circular wire loop of resistance R and radius a has its center at a distance d_0 ($d_0 > a$) from a long straight wire. The wire is in the plane of the loop. (See Fig. 1.) The current in the long wire is changing, $I = I(t)$. What is the current in the loop?

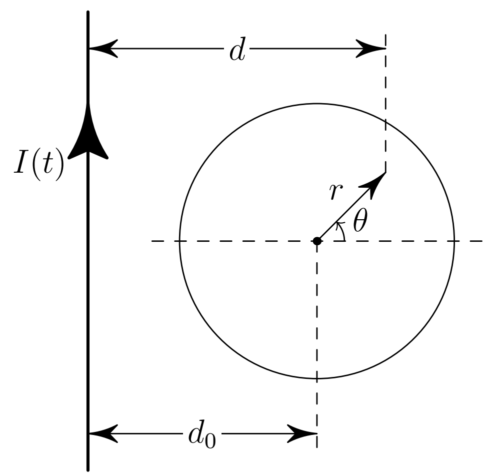


Figure 1: A long straight wire carrying current $I(t)$ and a circular wire loop with the center distance d_0 from the wire. The wire is in the plane of the loop

The magnitude of the emf induced in the loop is

$$\varepsilon = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\int B_n \,\mathrm{d}A\right|, \tag{1}$$

where Φ is the magnetic flux through the loop,

$$\Phi = \int B_n \,\mathrm{d}A, \tag{2}$$

B_n is the component of the magnetic field perpenducular to the plane of the loop; the integration is over the area of the loop. The magnetic field produced by the wire at a distance d from the wire,

$$B_n = \frac{\mu_0 I}{2\pi d}. \tag{3}$$

At a point inside the loop,

$$d = d_0 + r\cos\theta \tag{4}$$

and

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{5}$$

so that the flux through the loop is

$$\Phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{d_0 + r\cos\theta} = \frac{\mu_0 I}{2\pi}\int_0^{a}\phi(r)\,r\,\mathrm{d}r, \tag{6}$$

where

$$\phi(r) \equiv \int_0^{2\pi}\frac{\mathrm{d}\theta}{d_0 + r\cos\theta}. \tag{7}$$

Change the integration variable:

$$z = e^{i\theta}, \quad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{8}$$

In the z-plane the integration contour is the unit circle $|z| = 1$.

$$\phi(r) = -\frac{2i}{r}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2 + 2\frac{d_0}{r}z + 1} \tag{9}$$

The poles of the integrand are given by the roots of the quadratic polynomial in the denominator:

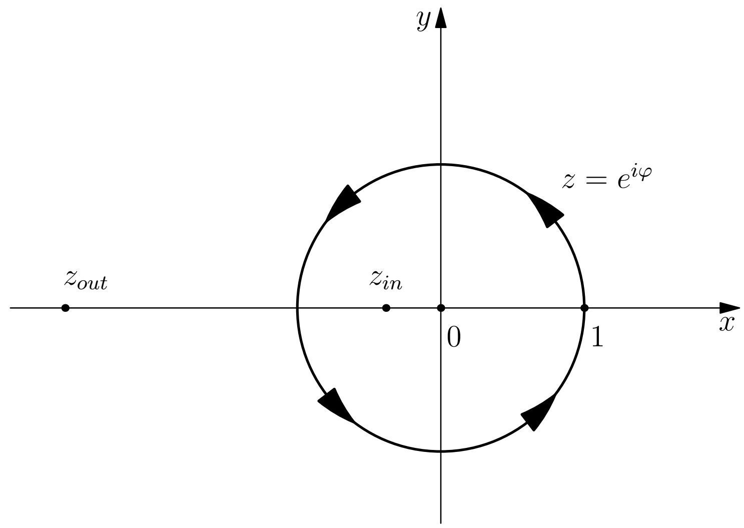
$$z^2 + 2\frac{d_0}{r}z + 1 = 0. \tag{10}$$

$$z_{in,out} = -\frac{d_0}{r} \pm \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}. \tag{11}$$

Since the loop is not crossing the wire, $d_0 > a \geq r$, i.e. $\frac{d_0}{r} > 1$. Therefore both z_{in} and z_{out} are real. Since $z_{in}z_{out} = 1$, only one root, the one with the smaller absolute value, is inside the integration contour:

$$z_{in} = -\frac{d_0}{r} + \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}. \tag{12}$$

Figure 2: Integration contour for Eq. (9). The poles of the integrand, z_{in} and z_{out}, are shown for $\frac{d_0}{r} = \frac{3}{2}$.



$$\phi(r) = 2\pi i \left(-\frac{2i}{r}\right) \mathrm{Res}\left(\frac{1}{(z-z_{in})(z-z_{out})}, z = z_{in}\right) \tag{13}$$

$$= \frac{4\pi}{r} \frac{1}{z_{in} - z_{out}} = \frac{2\pi}{r} \frac{1}{\sqrt{\left(\frac{d_0}{r}\right)^2 - 1}} = \frac{2\pi}{\sqrt{d_0^2 - r^2}}. \tag{14}$$

$$\Phi = \frac{\mu_0 I}{2\pi} \int_0^a \phi(r)\, r\, \mathrm{d}r = \mu_0 I \int_0^a \frac{r\, \mathrm{d}r}{\sqrt{d_0^2 - r^2}}. \tag{15}$$

Noting that

$$r\, \mathrm{d}r = \frac{1}{2} \mathrm{d}\left(r^2\right) = -\frac{1}{2} \mathrm{d}\left(d_0^2 - r^2\right) \tag{16}$$

and introducing new integration variable

$$u = d_0^2 - r^2, \quad d_0^2 - a^2 \le u \le d_0^2. \tag{17}$$

$$\Phi = \frac{\mu_0 I}{2} \int_{d_0^2 - a^2}^{d_0^2} \frac{\mathrm{d}u}{\sqrt{u}} = \mu_0 I \left.\sqrt{u}\right|_{u = d_0^2 - a^2}^{u = d_0^2} = \mu_0 I \left(d_0 - \sqrt{d_0^2 - a^2}\right). \tag{18}$$

We can check the result by looking at the limit $a \ll d_0$. We expect the flux to be approximately

$$\Phi \approx \frac{\mu_0 I}{2\pi d_0} \pi a^2 = \frac{\mu_0 I a^2}{2 d_0}. \tag{19}$$

Now, if we expand the square root in Equation (18), we get

$$\Phi = \mu_0 I d_0 \left(1 - \sqrt{1 - \left(\frac{a}{d_0}\right)^2}\right) \approx \mu_0 I d_0 \left[1 - 1 + \frac{1}{2}\left(\frac{a^2}{d_0}\right)\right] = \frac{\mu_0 I a^2}{2d_0} \tag{20}$$

as expected.

Finally, the current in the loop is

$$I = \frac{\varepsilon}{R} = \frac{\mu_0}{R}\left(d_0 - \sqrt{d_0^2 - a^2}\right)\frac{\mathrm{d}I}{\mathrm{d}t}. \tag{21}$$

References

[1] S. Lea, *Mathematics for Physicists*. Brooks Cole, 2003.