THE INTEGRAL THAT STUMPED FEYNMAN

Fall semester 2020

https://www.phys.uconn.edu/~rozman/Courses/P2400_20F/

Last modified: October 3, 2020

Q: In "Surely You're Joking, Mr. Feynman!," Nobel-prize winning physicist Richard Feynman said that he challenged his colleagues to give him an integral that they could evaluate with only complex methods that he could not do with real methods:

> One time I boasted, "I can do by other methods any integral anybody else needs contour integration to do."
>
> So Paul [Olum] puts up this tremendous damn integral he had obtained by starting out with a complex function that he knew the answer to, taking out the real part of it and leaving only the complex part. He had unwrapped it so it was only possible by contour integration! He was always deflating me like that. He was a very smart fellow.

Does anyone happen to know what this integral was?

A: I doubt that we will ever know the exact integral that vexed Feynman. Here is something similar to what he describes.

Suppose $f(z)$ is an analytic function on the unit disk. Then, by Cauchy's integral formula,

$$\oint_{\gamma} \frac{f(z)}{z}\,\mathrm{d}z = 2\pi i f(0), \tag{1}$$

where γ traces out the unit circle in a counterclockwise manner. Let

$$z = e^{i\phi}. \tag{2}$$

Then

$$\mathrm{d}z = ie^{i\phi}\mathrm{d}\phi = iz\,\mathrm{d}\phi \tag{3}$$

and

$$\int_0^{2\pi} f(e^{i\phi})\,\mathrm{d}\phi = 2\pi f(0). \tag{4}$$

Taking the real part of each side we find

$$\int\limits_0^{2\pi} \mathrm{Re}(f(e^{i\phi}))\,\mathrm{d}\phi = 2\pi\,\mathrm{Re}(f(0)). \tag{5}$$

(We could just as well take the imaginary part.) Clearly we can build some terrible integrals by choosing f appropriately.

For example, let

$$f(z) = \exp\left(\frac{2+z}{3+z}\right). \tag{6}$$

This is a mild choice compared to what could be done ... In any case, $f(z)$ is analytic on the unit disk. After some manipulations[1]

$$\mathrm{Re}\,(f(e^{i\phi})) = \exp\left(\frac{7+5\cos\phi}{10+6\cos\phi}\right)\cos\left(\frac{\sin\phi}{10+6\cos\phi}\right) \tag{7}$$

Applying Eq. (5), we find:

$$\int_0^{2\pi} \exp\left(\frac{7+5\cos\phi}{10+6\cos\phi}\right)\cos\left(\frac{\sin\phi}{10+6\cos\phi}\right) \mathrm{d}\phi = 2\pi e^{2/3}. \tag{8}$$

One way to check the result is to compare `NIntegrate[y, {f,0,2Pi}]` and `N[2Pi Exp[2/3]]`.

As another example, let

$$f(z) = \sin\left(z^2+1\right). \tag{9}$$

$f(z)$ is analytic on the unit disk. After similar manipulations as in the previous example,

$$\mathrm{Re}\,(f(e^{i\phi})) = \sin\left(2\cos^2(\phi)\right)\cosh\left(\sin(2\phi)\right) \tag{10}$$

Applying Eq. (5), we find:

$$\int_0^{2\pi} \sin\left(2\cos^2(\phi)\right)\cosh\left(\sin(2\phi)\right) \mathrm{d}\phi = 2\pi\sin(1). \tag{11}$$

One can check the result Eq. (11) comparing `NIntegrate[y, {f,0,2Pi}]` and `N[2Pi Sin[1]]`.

References

[1] Mathematics Stack Exchange. The integral that stumped Feynman. http://math.stackexchange.com/questions/253910/, 2012. URL http://math.stackexchange.com/questions/253910/. [Online; accessed 2015-02-19].

[1] `y = Simplify[ComplexExpand[Re[Exp[(2 + Exp[I f])/(3 + Exp[I f])]]]]`