The Gamma and Beta Functions

The Gamma and Beta Functions

We will now look at a use of double integrals outside of finding volumes. We will look at two of the most recognized functions in mathematics known as the Gamma Function and the Beta Function which we define below.

Definition: The **Gamma Function** is defined as the single variable function $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t} \: dt$ for $x > 0$, and the **Beta Function** is defined as the two variable function $B(x, y) = \int_0^1 t^{s-1}(1-t)^{y-1} \: dt$ for $x, y > 0$.

The Gamma function is important as it is an extension to the factorial function $f(n) = n!$ for all $n \in \mathbb{N}$, and thus, $\Gamma(n+1) = n!$. To show this, let $n \in \mathbb{N}$. Then:

$$\Gamma(n+1) = \int_0^{\infty} t^{(n+1)-1} e^{-t} \: dt = \int_0^{\infty} t^n e^{-t} \: dt \tag{1}$$

Now let $u = t^n$ and $dv = e^{-t} \: dt$. Then $du = nt^{n-1} \: dt$ and $v = -e^{-t}$, so in applying the technique of integration by parts, we have that:

$$\Gamma(n+1) = -t^ne^{-t} \biggr \rvert_0^{\infty} + \int_0^{\infty} nt^{n-1}e^{-t} \: dt = \lim_{b \to \infty} -b^ne^{-b} + n \int_0^{\infty} t^{n-1}e^{-t} \: dt = n \int_0^{\infty} t^{n-1} e^{-t} \: dt \tag{2}$$

If we continue onward by applying integration by parts again and again, then we eventually have that:

$$\Gamma(n+1) = n(n-1)...(2) \int_0^{\infty} e^{-t} \: dt = n! \lim_{b \to \infty} \left (-e^{-t} + 1 \right) \: dt = n! \tag{3}$$

Another nice property of the Gamma function is that for any $x > 0$, we have that $\Gamma(x + 1) = x\Gamma(x)$ since (by applying integration by parts again):

$$\Gamma(x+1) = \int_0^{\infty} t^xe^{-t} \: dt = -t^xe^{-t} \biggr \rvert_0^{\infty} + \int_0^{\infty} xt^{x-1}e^{-t} \: dt = \underbrace{\lim_{b \to \infty} -t^xe^{-t}}_{=0} + x \underbrace{\int_0^{\infty} t^{x-1}e^{-t} \: dt}_{= \Gamma(x)} = x\Gamma(x) \tag{4}$$

Another neat property of the Gamma function is that $\Gamma \left (\frac{1}{2} \right) = \sqrt{\pi}$. This is interesting since our integral does not contain a term that one would naturally assume for the value π to arise in. Verifying this property is relatively easy once we make an appropriate substitution. Let $t = s^2$. Then $dt = 2s \: ds$ and so we have that:

$$\Gamma(x) = \int_0^{\infty} (s^2)^{x-1} e^{-s^2} 2s \: ds = 2 \int_0^{\infty} s^{2x-1}e^{-x^2} \: ds \tag{5}$$

Now plugging in $x = \frac{1}{2}$ and recalling from the Evaluating Double Integrals in Polar Coordinates page that $\int_{-\infty}^{\infty} e^{-x^2} \: dx = \sqrt{\pi}$ and so $\int_0^{\infty} e^{-x^2} \: dx = \frac{\sqrt{\pi}}{2}$, we have that:

$$\Gamma \left (\frac{1}{2} \right) = 2 \int_0^{\infty} e^{-x^2} \: ds = \sqrt{\pi} \tag{6}$$

Now the Beta function is also a very interesting function in how it relates to the Gamma function. One such property of the Beta function is that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for $x, y > 0$. To show this, we will need to make a substitution for the Beta function. Let $t = \cos^2 \theta$. Then $dt = -2 \sin \theta \cos \theta \: d\theta$. Now we need to find the bounds of integration. Note that $t = \cos^2 \theta$ with $t = 0$ implies that $\theta = \frac{\pi}{2}$, and $t = \cos^2 \theta$ with $t = 1$ implies that $\theta = 0$, and so:

$$B(x, y) = -2 \int_{\pi/2}^0 (\cos^2 \theta)^{x-1} (1 - \cos^2 \theta)^{y-1} \sin \theta \cos \theta \: d\theta = -2 \int_{\pi/2}^0 \cos^{2x-1} \theta \sin^{2y-1} \theta \: d\theta = 2 \int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta \: d\theta \tag{7}$$

Now we will proceed to show that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$:

(8)

$$\Gamma(x)\Gamma(y) = \left(2\int_0^\infty s^{2x-1}e^{-s^2}\,ds\right)\left(2\int_0^\infty m^{2y-1}e^{-m^2}\,dm\right)$$

Now let $D = \{(s,m) \in \mathbb{R}^2 : 0 \leq s \leq \infty, 0 \leq m \leq \infty\}$. Then the integrals above can be condensed into a double integral as:

$$= 4\iint_Q s^{2x-1}m^{2y-1}e^{-s^2-m^2}\,ds\,dm \tag{9}$$

Now we will convert the double integral by using polar coordinates. Let $s = r\cos\theta$ and $m = r\sin\theta$. Then $-s^2 - m^2 = -r^2$. Furthermore $D = \left\{(r,\theta) : 0 \leq r \leq \infty, 0 \leq \theta \leq \frac{\pi}{2}\right\}$ and so:

$$= 4\int_0^{\pi/2}\int_0^\infty r^{2x-1}\cos^{2x-1}\theta r^{2y-1}\sin^{2y-1}\theta e^{-r^2}\,r\,dr\,d\theta \tag{10}$$

$$= \underbrace{\left(2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,d\theta\right)}_{=B(x,y)}\underbrace{\left(2\int_0^\infty r^{2(x+y)-1}e^{-r^2}\,dr\right)}_{=\Gamma(x+y)} = B(x,y)\Gamma(x+y)$$

Thus we have that $\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)$ so $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. As you can see, the use of double integrals came in handy in deriving this fact.

Powered by Wikidot.com

Help | Terms of Service | Privacy | Report a bug | Flag as objectionable

Unless otherwise stated, the content of this page is licensed under Creative Commons Attribution-ShareAlike 3.0 License