Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form. Write your name on the problems page and enclose it together with your solutions. Use **only** the methods we introduced in class.

Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	15	10	10	40	10	15	100
Score:							

The problem:

This problem concerns the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - x - 2\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \tag{1}$$

where ε is a large parameter, $\varepsilon \gg 1$. Follow the procedure described in the course notes on relaxation oscillations, *Singular perturbation theory*, to find the period, $T(\varepsilon)$, and "amplitude" of the limit cycle of this equation.

Assume that Eq. (1) is already written in dimensionless form.

Consult the numerical solution of Eq. (1) – see Fig 1.

1. (15 points) Rewrite Eq. (1) to use a small parameter $\nu = \frac{1}{\varepsilon}$. Scale the time variable $t = \nu^{\alpha}\,\tau$ and use the method of dominant balance to identify two solvable differential equations that describe the system's "slow" and "fast" motion.

2. (10 points) Integrate the "slow" differential equation.

3. (10 points) Reduce the "fast" differential equations (which is a second order differential equation) to a first order differential equation. (There is no need to completely integrate the equation.) Note that the "new fast" equation must include one integration constant.

4. Proceed to the matching of the solutions for the slow and the fast regimes.

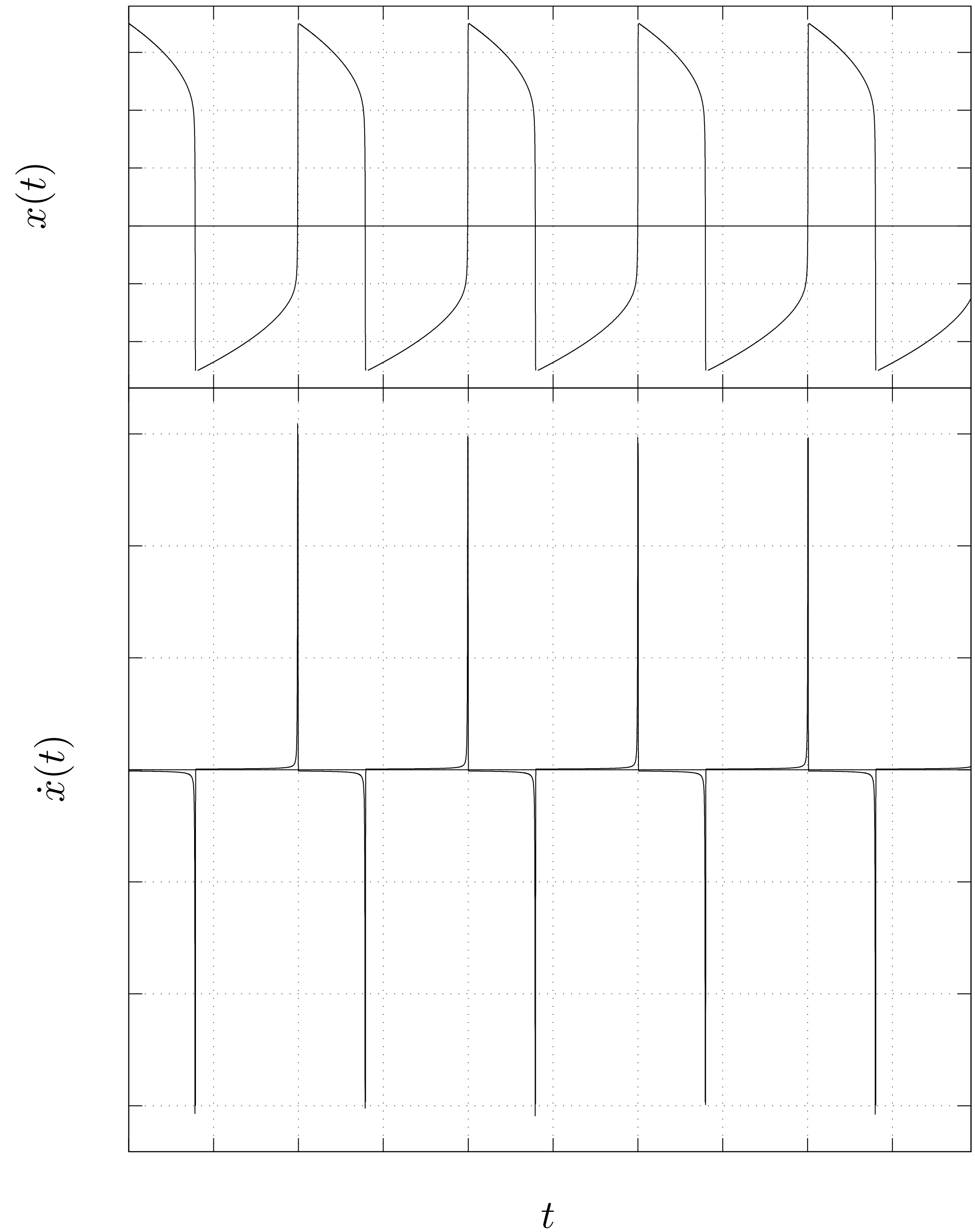

Figure 1: Numerical solution of Eq. (1) for $\varepsilon = 10$.

(a) (5 points) Start by following the oscillator as it moves from its largest positive displacement. Use the "slow" differential equation to identify the point of the transition from the slow to the fast regimes.

(b) (5 points) Use the x coordinate of the transition to find the integration constant in the "new fast" equation.

(c) (5 points) Factor the right hand side of the "new fast" equation and determine x coordinate of the transition from the fast to the slow motion. You found the largest in absolute value negative displacement of the oscillator!

(d) (5 points) You now start following the oscillator as it moves toward positive displacements. Use the "slow" differential equation to identify the point of transition from the slow to the fast regimes.

(e) (5 points) You now have the coordinates of the beginning and the end of the slow motion for **negative** x. Use the solution of the "slow" differential equation to determine the duration of the "slow negative" part.

(f) (5 points) Use the x coordinate of the transition to find another integration constant in the "new fast" equation.

(g) (5 points) Factor the right hand side of the "new fast" equation and determine x coordinate of the transition from the fast to the slow motion. You found the largest positive displacement of the oscillator!

(h) (5 points) You now have the coordinates of the beginning and the end of the slow motion for **positive** x. Use the solution of the "slow" differential equation to determine the duration of the "slow positive" part.

Neglect the duration of the fast part of the limit cycle. The total duration of the slow parts is approximately the period of the oscillations.

5. (10 points) Solve Eq. (1) numerically for a sufficiently large value of ε of your choice. As the initial conditions, choose the maximal positive displacement in the limit cycle and zero velocity. Plot your solution for several periods of oscillations. Estimate the period of the limit cycle from your graph. Compare the result of the numerical calculations and the theoretical analysis.

6. (15 points) Provide a clear *self-contained* description of the problem you solved and the solution steps.