ELLIPTIC INTEGRALS MEET FALLING CHAIN [*]

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1 Introduction

A chain of length L is initially suspended with both ends at the same vertical elevation and then one end is released (see Fig. 1). What is the time T it takes the free end to reach the bottom?

A solution pictures the links on the free side falling independently with acceleration of gravity g. When a link reaches bottom, it undergoes an inelastic collision with the fixed side and is brought to rest. This gives rise to a tension at the bottom on the fixed side but not on the free one. Within this picture the time for the free end to reach the bottom is simply the time of the free fall over the hight L: $\sqrt{\frac{2L}{g}}$. However, this answer is very different from real chain's behavior.

[*] The title inspired by: P. Nahin, *Inside Interesting Integrals*, Springer, 2015.

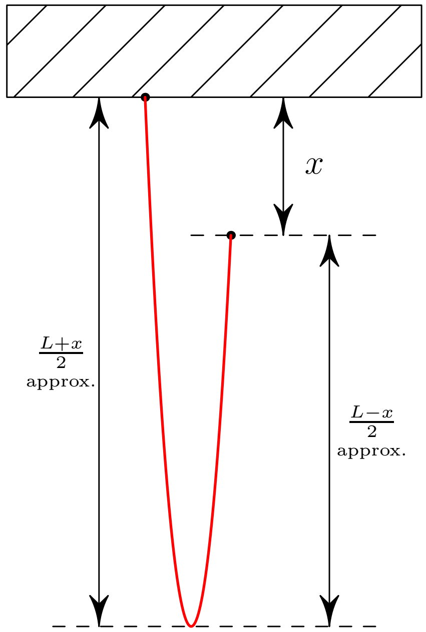

Figure 1: A chain is fixed at one end and the other end is released from the same elevation.

2 Elliptic integrals

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\varphi'}{\sqrt{1-k^2\sin^2(\varphi')}}, \tag{1}$$

$$E(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2(\varphi')}\, d\varphi' \tag{2}$$

are the elliptic integrals of the first and second kind, respectively; $0 \le k \le 1$; the constant k is called the modulus. When $\varphi = \frac{\pi}{2}$ the integrals are called complete, while the integrals are called incomplete otherwise. Except for the two special cases of $k = 0$ and $k = 1$, $F(k,\varphi)$ and $E(k,\varphi)$ are not expressible in terms of elementary functions.

3 Energy-conserving chain

Experimental observations show that real chains act much more like a perfectly flexible, inextensible rope, which when falls conserves the energy.

For moderate initial distance between the ends of the chain (the *width*), the potential energy is a very slowly varying function of width, and the contribution to the kinetic energy from the horizontal motion is a small fraction of that due to the vertical motion. Thus, if the initial width is not too large, we expect to be able to approximate the motion by the limiting one-dimensional motion of a 'zero-width' chain.

In this limiting case, the center of mass of the left-hand-side of the rope is $\frac{1}{2}\frac{L+x}{2}=\frac{L+x}{4}$ below the fixed end, while the center of mass of the right-hand-side of the rope is $x+\frac{1}{2}\frac{L-x}{2}=\frac{L+3x}{4}$ below the fixed end. Furthermore, the mass of the left-hand-side of the rope is $\mu\frac{L+x}{2}$, where μ is the linear density of the chain (i.e. the the mass of the chain per unit length); the mass of the right-hand-side of the rope is $\mu\frac{L-x}{2}$. Therefore, the potential energy, relative to the fixed end, is

$$\begin{aligned} V(x) &= -\mu g\frac{L+x}{4}\frac{L+x}{2}-\mu g\frac{L+3x}{4}\frac{L-x}{2} \\ &= -\frac{\mu g}{8}\left(L^2+2Lx+x^2+L^2+3Lx-Lx-3x^2\right) \\ &= -\frac{\mu g}{4}\left(L^2+2Lx-x^2\right), \end{aligned} \tag{3}$$

where g is the acceleration of gravity.

The kinetic energy of the chain is due to the motion of its right hand side:

$$T(x,\dot{x})=\frac{1}{2}\mu\frac{L-x}{2}\dot{x}^2, \tag{4}$$

since every moving element of the chain moves with the same velocity - the velocity of the free end $\dot{x}$.

Conservation of energy then gives the speed at of the free end in terms of its position x:

$$\frac{1}{2}\mu\frac{L-x}{2}\dot{x}^2-\frac{\mu g}{4}\left(L^2+2Lx-x^2\right)=-\frac{\mu g}{4}L^2, \tag{5}$$

where the expression on the right is the initial potential energy of the chain.

Rearranging terms in Eq. (5), obtain:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{\frac{gx(2L-x)}{L-x}}. \tag{6}$$

Separating variables in the ordinary differential equation Eq. (6),

$$\mathrm{d}t = \sqrt{\frac{L-x}{gx(2L-x)}}\,\mathrm{d}x. \tag{7}$$

Integrating both sides of Eq. (5), on the left with respect to x, from 0 to L, and on the left with respect to t, from 0 to T, obtain:

$$T = \int_0^L \sqrt{\frac{L-x}{gx(2L-x)}}\,\mathrm{d}x = \sqrt{\frac{2L}{g}}\left\{\frac{1}{\sqrt{2}}\int_0^1 \sqrt{\frac{1-u}{u(2-u)}}\,\mathrm{d}u\right\} = \sqrt{\frac{2L}{g}}\,\gamma. \tag{8}$$

In the second equality we wrote the dimensional factor which is equal to the time of the free fall over the heigth L, and introduced the dimensionless integration variable $u = \frac{x}{L}$, $0 \le u \le 1$.

$$\gamma = \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\frac{1-u}{u(2-u)}}\,\mathrm{d}u. \tag{9}$$

$$u = \sin^2\varphi, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad \mathrm{d}u = 2\sin\varphi\cos\varphi\,\mathrm{d}\varphi. \tag{10}$$

$$\gamma = \sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin^2\varphi}{\sin^2\varphi\left(2-\sin^2\varphi\right)}}\,\sin\varphi\cos\varphi\,\mathrm{d}\varphi \tag{11}$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{1-\sin^2\varphi}{\sqrt{\left(2-\sin^2\varphi\right)}}\,\mathrm{d}\varphi = 2\int_0^{\frac{\pi}{2}} \frac{1-\frac{1}{2}\sin^2\varphi-\frac{1}{2}}{\sqrt{\left(1-\frac{1}{2}\sin^2\varphi\right)}}\,\mathrm{d}\varphi \tag{12}$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{2}\sin^2\varphi}\,\mathrm{d}\varphi - \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}} \tag{13}$$

$$= 2E\left(\frac{1}{\sqrt{2}}\right) - F\left(\frac{1}{\sqrt{2}}\right) \approx 0.847. \tag{14}$$

This is shorter than the free-fall time by about 15%. Thus the acceleration is greater than g, and the velocity is greater than the free fall $g\,t$.

The reason for this behavior is that there is nonzero tension T in the chain on the free side of the bend, which pulls the chain down in addition to gravity.

References

[1] M. G. Calkin and R. H. March, *The Dynamics of a Falling Chain. I*, Am. J. Phys., **57**(2), February 1989,pp. 154–157.