Method of stationary phase

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There is an immediate generalization of the Laplace integrals

$$\int_a^b f(t)e^{x\phi(t)}\,\mathrm{d}t \tag{1}$$

which we obtain by allowing the function $\phi(t)$ in Eq. (1) to be complex. We may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses nontrivial problems. We consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$ where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,\mathrm{d}t \tag{2}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral.

The method of stationary phase gives the leading asymptotic behavior of generalized Fourier integrals having stationary points, $\psi' = 0$. This method is similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval surrounding the stationary points of ψ.

Recall that

$$\int\limits_{-\infty}^{\infty} e^{\pm iu^2}\,\mathrm{d}u = \sqrt{\pi}\, e^{\pm i\frac{\pi}{4}}, \qquad \int\limits_{-\infty}^{\infty} e^{\pm i\lambda u^2}\,\mathrm{d}u = \sqrt{\frac{\pi}{\lambda}}\, e^{\pm i\frac{\pi}{4}}. \tag{3}$$

Example 1. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int\limits_{-3}^{4} \cos\left(\lambda \sinh^2(x)\right)\, \sqrt{1+x^2}\,\mathrm{d}x. \tag{4}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ are important,

$$\sinh x \sim x, \tag{5}$$

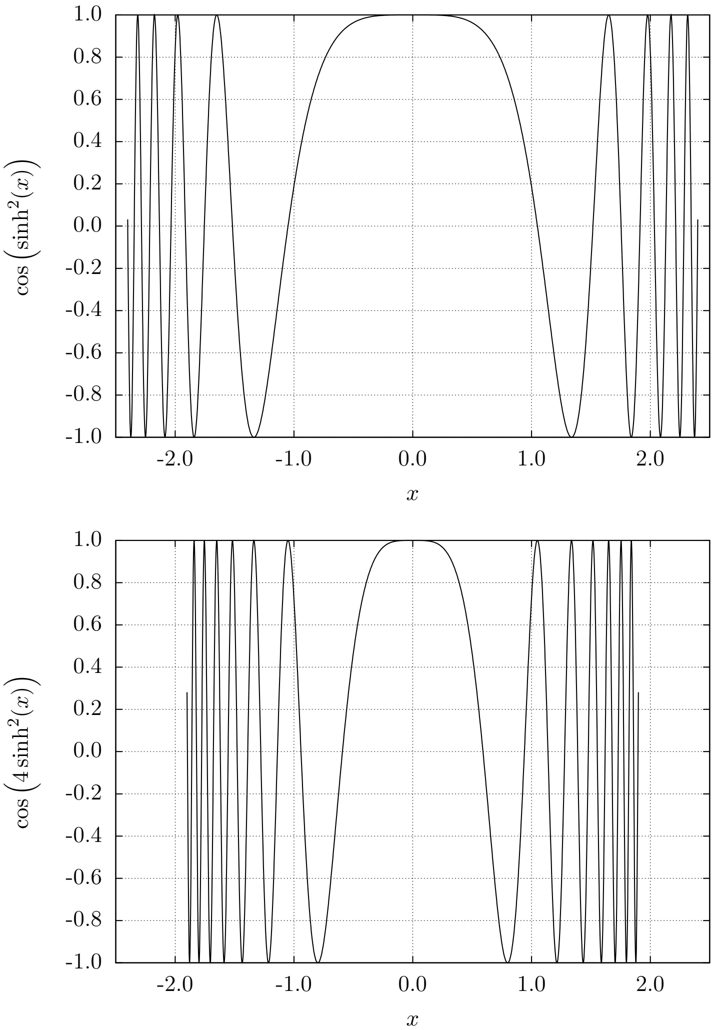

Figure 1: The graphs of the oscillating factor, $\cos\left(\lambda\sinh^2(x)\right)$ in Eq. (4), for $\lambda = 1, 4$.

$$\cos\left(\lambda\sinh^2(x)\right) \sim \cos\left(\lambda x^2\right) = \operatorname{Re} e^{i\lambda x^2} \tag{6}$$

$$\sqrt{1+x^2} \sim 1. \tag{7}$$

$$I(\lambda) \sim \operatorname{Re}\int_{-3}^{4} e^{i\lambda x^2}\,\mathrm{d}x \sim \operatorname{Re}\int_{-\infty}^{\infty} e^{i\lambda x^2}\,\mathrm{d}x. \tag{8}$$

New integration variable,

$$u^2 = \lambda x^2 \quad\longrightarrow\quad x^2 = \frac{u^2}{\lambda} \quad\longrightarrow\quad x = \frac{u}{\sqrt{\lambda}} \quad\longrightarrow\quad \mathrm{d}x = \frac{1}{\sqrt{\lambda}}\,\mathrm{d}u. \tag{9}$$

$$I(\lambda) \sim \operatorname{Re}\frac{1}{\sqrt{\lambda}}\underbrace{\int_{-\infty}^{\infty} e^{iu^2}\,\mathrm{d}u}_{\sqrt{\pi}e^{i\frac{\pi}{4}}} = \sqrt{\frac{\pi}{\lambda}}\,\underbrace{\operatorname{Re}\left(e^{i\frac{\pi}{4}}\right)}_{\frac{1}{\sqrt{2}}} = \boxed{\sqrt{\frac{\pi}{2\lambda}}} \tag{10}$$

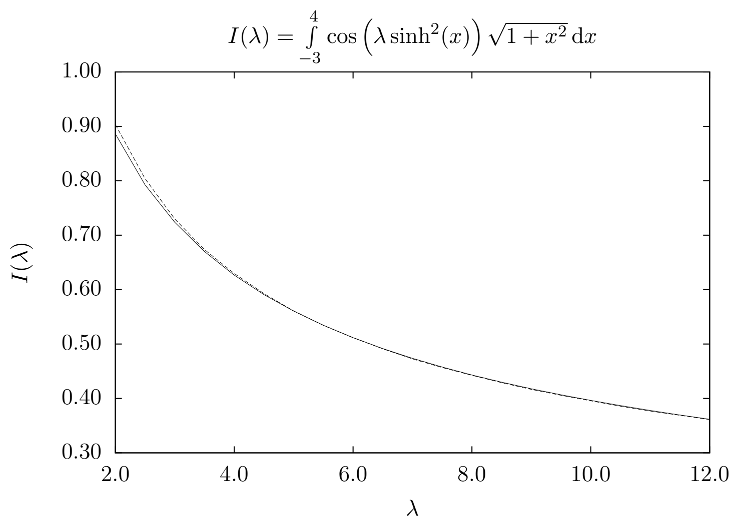

Figure 2: Asymptotics Eq. (10) (solid line) compared to numerically evaluated Eq. (4) (dashed line) for $2 \le \lambda \le 12$.

Example 2. Find the leading term of the asymptotics of the Bessel function $J_0(x)$ for $x \to \infty$:

$$J_0(x) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x \cos\theta\right)\, d\theta \tag{11}$$

Bessel function $J_0(x)$ is a solution of the following second order linear differential equation:

$$x y'' + y' + x y = 0. \tag{12}$$

Let's show first that Eq. (11) is indeed a solution of Eq. (12).

$$\frac{d}{dx} J_0(x) = -\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(x \cos\theta\right) \cos\theta\, d\theta, \tag{13}$$

$$\frac{d^2}{dx^2} J_0(x) = -\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x \cos\theta\right) \cos^2\theta\, d\theta. \tag{14}$$

$$\begin{aligned}
x\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}J_0(x)+J_0(x)\right) &= \frac{x}{\pi}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\cos^2\theta\right)\,\cos\left(x\cos\theta\right)\,\mathrm{d}\theta \\
&= \frac{x}{\pi}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\,\cos\left(x\cos\theta\right)\,\mathrm{d}\theta \\
&= -\frac{1}{\pi}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\,\cos\left(x\cos\theta\right)\,\mathrm{d}(x\cos\theta) \\
&= -\frac{1}{\pi}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\,\mathrm{d}\left(\sin\left(x\cos\theta\right)\right) \\
&= -\frac{1}{\pi}\sin\theta\,\sin\left(x\cos\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{1}{\pi}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\left(x\cos\theta\right)\,\cos\theta\,\mathrm{d}\theta \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}J_0(x),
\end{aligned} \tag{15}$$

which is indeed in agreement with Eq. (12).

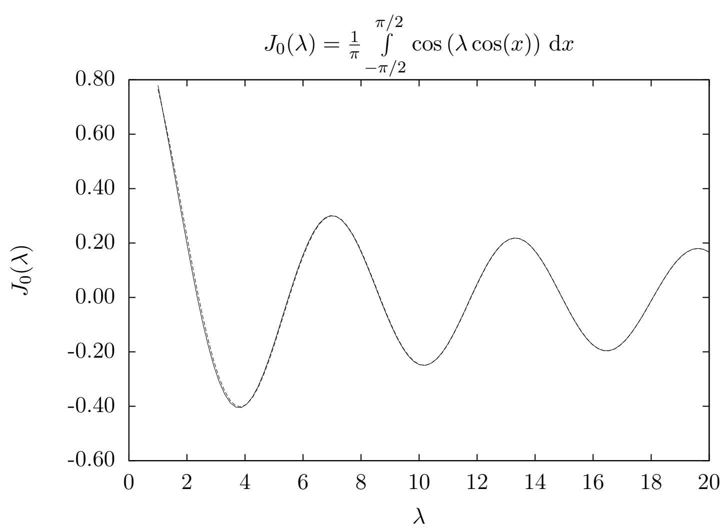

Figure 3: Asymptotics Eq. (18) (solid line) compared to numerically evaluated Eq. (11) (dashed line) for $1 \le x \le 20$.

Let's rewrite integral Eq. (11) in the exponential form:

$$J_0(x)=\frac{1}{\pi}\,\mathrm{Re}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{ix\cos\theta}\mathrm{d}\theta. \tag{16}$$

The stationary point of the phase factor is at $\theta = 0$. Only small θ contribute to the integral. Therefore.

$$\cos\theta \approx 1-\frac{\theta^2}{2}. \tag{17}$$

$$
\begin{aligned}
J_0(x) &\sim \frac{1}{\pi}\,\mathrm{Re}\,e^{ix}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\frac{x\theta^2}{2}}\mathrm{d}\theta \sim \frac{1}{\pi}\sqrt{\frac{2}{x}}\,\mathrm{Re}\,e^{ix}\int_{-\infty}^{\infty} e^{-i\frac{x}{2}\theta^2}\,\mathrm{d}\left(\sqrt{\frac{x}{2}}\,\theta\right) \\
&= \frac{1}{\pi}\sqrt{\frac{2}{x}}\,\mathrm{Re}\left(e^{ix}\,\sqrt{\pi}e^{-i\frac{\pi}{4}}\right) = \boxed{\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right)}
\end{aligned}
\tag{18}
$$

Integration by parts

If $\psi(t)$ in the integral Eq. (2) has no stationary point, $\psi'(t)=0$, in the integration range $[a,b]$, the method of stationary phase is not applicable. In this case a simple integration by parts gives the leading asymptotic behaviour.

$$
\begin{aligned}
I(x) &= \int_a^b f(t)e^{ix\psi(t)}\,\mathrm{d}t = \frac{1}{ix}\int_a^b \frac{f(t)}{\psi'(t)}\,\mathrm{d}\left(e^{ix\psi(t)}\right) \\
&= \frac{1}{ix}\frac{f(t)}{\psi'(t)}\,e^{ix\psi(t)}\Big|_a^b - \frac{1}{ix}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\psi'(t)}\right)e^{ix\psi(t)}\,\mathrm{d}t.
\end{aligned}
\tag{19}
$$

The integral on the right vanishes more rapidly than $1/x$ (Riemann–Lebesgue lemma). Therefore,

$$
I(x) \sim \frac{1}{ix}\frac{f(t)}{\psi'(t)}\,e^{ix\psi(t)}\Big|_a^b \tag{20}
$$

as $x\to\infty$.

Example 3.

$$
I(x) = \int_0^1 \frac{\cos(xt)}{1+t}\,\mathrm{d}t = \mathrm{Re}\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t. \tag{21}
$$

Integrating the last integral by parts, we obtain

$$
\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t = \frac{1}{ix}\int_0^1 \frac{1}{1+t}\,\mathrm{d}\left(e^{ixt}\right) = \frac{1}{ix}\left(\frac{e^{ix}}{2}-1\right) + \frac{1}{ix}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,\mathrm{d}t. \tag{22}
$$

The last term on the right is $\sim x^{-2}$ (see below), therefore the leading term in the approximation of Eq. (21) when $x\to\infty$ is

$$
I(x) \approx \mathrm{Re}\left\{\frac{1}{ix}\left(\frac{e^{ix}}{2}-1\right)\right\} = \frac{\sin(x)}{2x}. \tag{23}
$$

We can continue the integration by parts of the integral in the right hand side of Eq. (22):

$$
\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,\mathrm{d}t = \frac{1}{ix}\int_0^1 \frac{1}{(1+t)^2}\,\mathrm{d}\left(e^{ixt}\right) = \frac{1}{ix}\left(\frac{e^{ix}}{4}-1\right) + \frac{2}{ix}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,\mathrm{d}t. \tag{24}
$$

Thus,

$$
\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t = \frac{1}{ix}\left(\frac{e^{ix}}{2}-1\right) - \frac{1}{x^2}\left(\frac{e^{ix}}{4}-1\right) - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,\mathrm{d}t. \tag{25}
$$

The last term in the right hand side of Eq. (25) is of order x^{-3} and can be neglected, therefore

$$I(x) \approx \mathrm{Re}\left\{\frac{1}{ix}\left(\frac{e^{ix}}{2}-1\right)-\frac{1}{x^2}\left(\frac{e^{ix}}{4}-1\right)\right\}=\frac{\sin(x)}{2x}-\frac{1}{x^2}\left(\frac{\cos(x)}{4}-1\right) \tag{26}$$

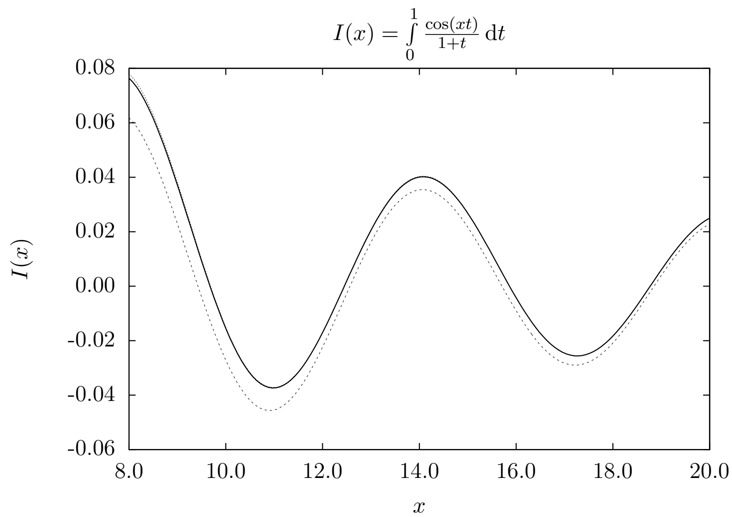

Figure 4: Asymptotics Eq. (23) (dashed line) and Eq. (26) (dotted line) compared to numerically evaluated Eq. (21) (solid line) for $8 \le x \le 20$.

References

[1] Lorella M. Jones. *An introduction to mathematical methods of physics*. Benjamin Cummings, 1979.

[2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.