

Summation of series: Sommerfeld-Watson transformation

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Sommerfeld-Watson transformation, also called Watson transformation, is a technique for converting series into contour integrals using the Cauchy residue theorem. The technique was known already to Cauchy[1]. It got more attention and development in the early twentieth century in connection with the problem of radio waves propagation around the Earth[2, 3], and in mid-twentieth century in connection with the study of the analytic properties of scattering amplitude in quantum mechanics.

Below we assume that $f(z)$ is a function which is analytic at the integers $z = n$, $n = 0, \pm 1, \pm 2, \dots$ and tends to zero at least as fast as

$$\frac{1}{|z|^2} \quad \text{as} \quad |z| \rightarrow \infty. \quad (1)$$

Otherwise $f(z)$ is an arbitrary function.

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Our goal is to derive a usable expression for the following sum:

$$S = \sum_{n=-\infty}^{\infty} f(n) \quad (2)$$

Let's consider the function

$$F(z) = \pi f(z) \frac{\cos(\pi z)}{\sin(\pi z)}. \quad (3)$$

This function has simple poles at $z = n$, $n = 0, \pm 1, \pm 2, \dots$ with residues

$$\begin{aligned} \text{Res}(F(z), z = n) &= \lim_{z \rightarrow n} \frac{\pi f(z) \cos(\pi z)}{\frac{d}{dz} \sin(\pi z)} = \lim_{z \rightarrow n} \frac{\pi f(z) \cos(\pi z)}{\pi \cos(\pi z)} = f(n). \\ \text{Res}(F(z), z = n) &= f(n). \end{aligned} \quad (4)$$

The poles of $f(z)$, $z = z_i$, are also poles of $F(z)$ that are different from the poles at $z = n$.

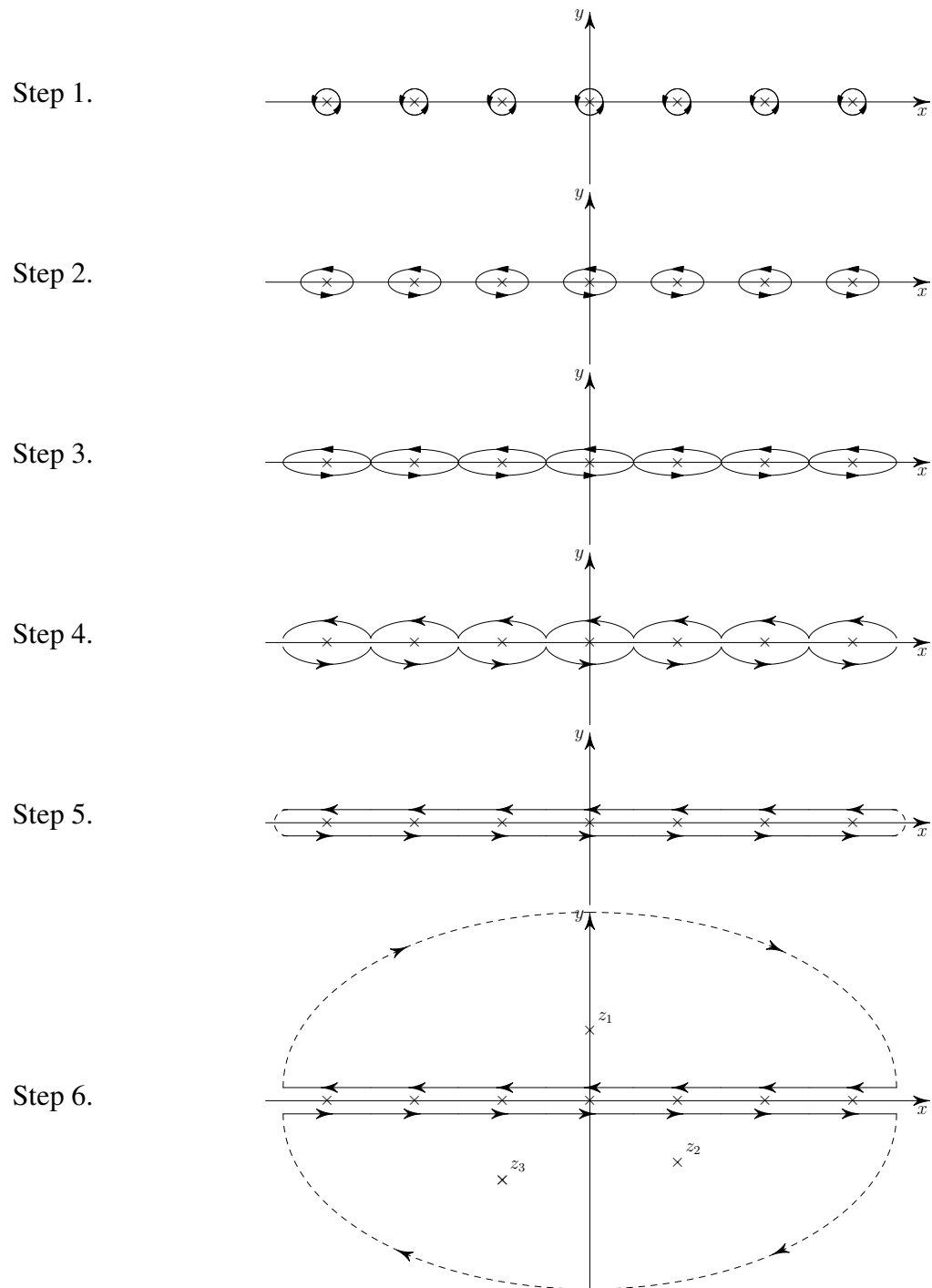


Figure 1: Transformation of the integration contour for Eq. (2)

The integral of $F(z)$ over the circle centered at the origin and of radius $R \rightarrow \infty$ is zero due to conditions Eq. (1).

$$\oint_C F(z) dz = 0. \quad (5)$$

On the other hand

$$\oint_C F(z) dz = 2\pi i \left[\sum_{n=-\infty}^{\infty} \text{Res}(F(z), z = n) + \sum_i \text{Res}(F(z), z = z_i) \right]. \quad (6)$$

Comparing Eq. (6) and Eq. (5), and using Eq. (4) we arrive at the following result:

$$\sum_{n=-\infty}^{\infty} f(n) = - \sum_i \text{Res}(F(z), z = z_i), \quad (7)$$

where the summation on the right is over the poles of $f(z)$, and $F(z)$ is given by Eq. (3).

Example 1. Let's evaluate the following sum:

$$S(a) = \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2}, \quad (8)$$

where a is a real positive parameter.

1. Convert the summation over the integer n from $[0, \infty)$ to $(-\infty, \infty)$. Since the terms in the sum are even functions of n ,

$$S(a) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{2a^2}, \quad (9)$$

where the last term on the right of Eq. (9) takes into account that the term with $n = 0$ appears in the summation only once.

2. The sum in Eq. (9) has the form of Eq. (2). Therefore we can use the developed technique summarized in Eqs. (7), (3). Here

$$f(z) = \frac{1}{z^2 + a^2}. \quad (10)$$

$f(z)$ has two simple poles at

$$z_{1,2} = \pm ia. \quad (11)$$

$$F(z) = \frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}. \quad (12)$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\text{Res} \left(\frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = ia \right) - \text{Res} \left(\frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = -ia \right). \quad (13)$$

The first term on the right,

$$\text{Res} \left(\frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = ia \right) = \lim_{z \rightarrow ia} \left[\frac{\pi(z - ia)}{(z + ia)(z - ia)} \frac{\cos(\pi z)}{\sin(\pi z)} \right] = \frac{\pi}{2ia} \frac{\cos(i\pi a)}{\sin(i\pi a)}. \quad (14)$$

$$\cos(i\pi a) = \frac{1}{2} (e^{i i \pi a} + e^{-i i \pi a}) = \frac{1}{2} (e^{-\pi a} + e^{\pi a}) = \cosh(\pi a). \quad (15)$$

$$\sin(i\pi a) = \frac{1}{2i} (e^{i i \pi a} - e^{-i i \pi a}) = \frac{1}{2i} (e^{-\pi a} - e^{\pi a}) = i \sinh(\pi a). \quad (16)$$

Thus,

$$\text{Res} \left(\frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = ia \right) = \frac{\pi}{2ia} \frac{\cosh(\pi a)}{i \sinh(\pi a)} = -\frac{\pi}{2a} \frac{\cosh(\pi a)}{\sinh(\pi a)}. \quad (17)$$

Similarly,

$$\text{Res} \left(\frac{\pi}{z^2 + a^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = -ia \right) = \frac{\pi}{2ia} \frac{\cosh(\pi a)}{i \sinh(\pi a)} = -\frac{\pi}{2a} \frac{\cosh(\pi a)}{\sinh(\pi a)}. \quad (18)$$

Therefore, from Eqs. (13), (17), and (18)

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \frac{\cosh(\pi a)}{\sinh(\pi a)}, \quad (19)$$

and, finally, from Eq. (9),

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \frac{\cosh(\pi a)}{\sinh(\pi a)}. \quad (20)$$

Example 2. Basel problem The Basel problem is a problem in mathematical analysis, first posed in 1644 and solved by Euler in 1734. Since the problem had withstood the attacks of the leading mathematicians of the day for almost hundred years, Euler's solution brought him immediate fame when he was twenty-eight. The problem is named after Basel, the hometown of Euler.

The Basel problem asks for the sum of the following infinite series:

$$B = \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (21)$$

1. Convert the summation over the integer n from $[1, \infty)$ to $(-\infty, \infty)$, with the exclusion of the term corresponding to $n = 0$. Since the terms in the sum are even functions of n ,

$$B = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2}. \quad (22)$$

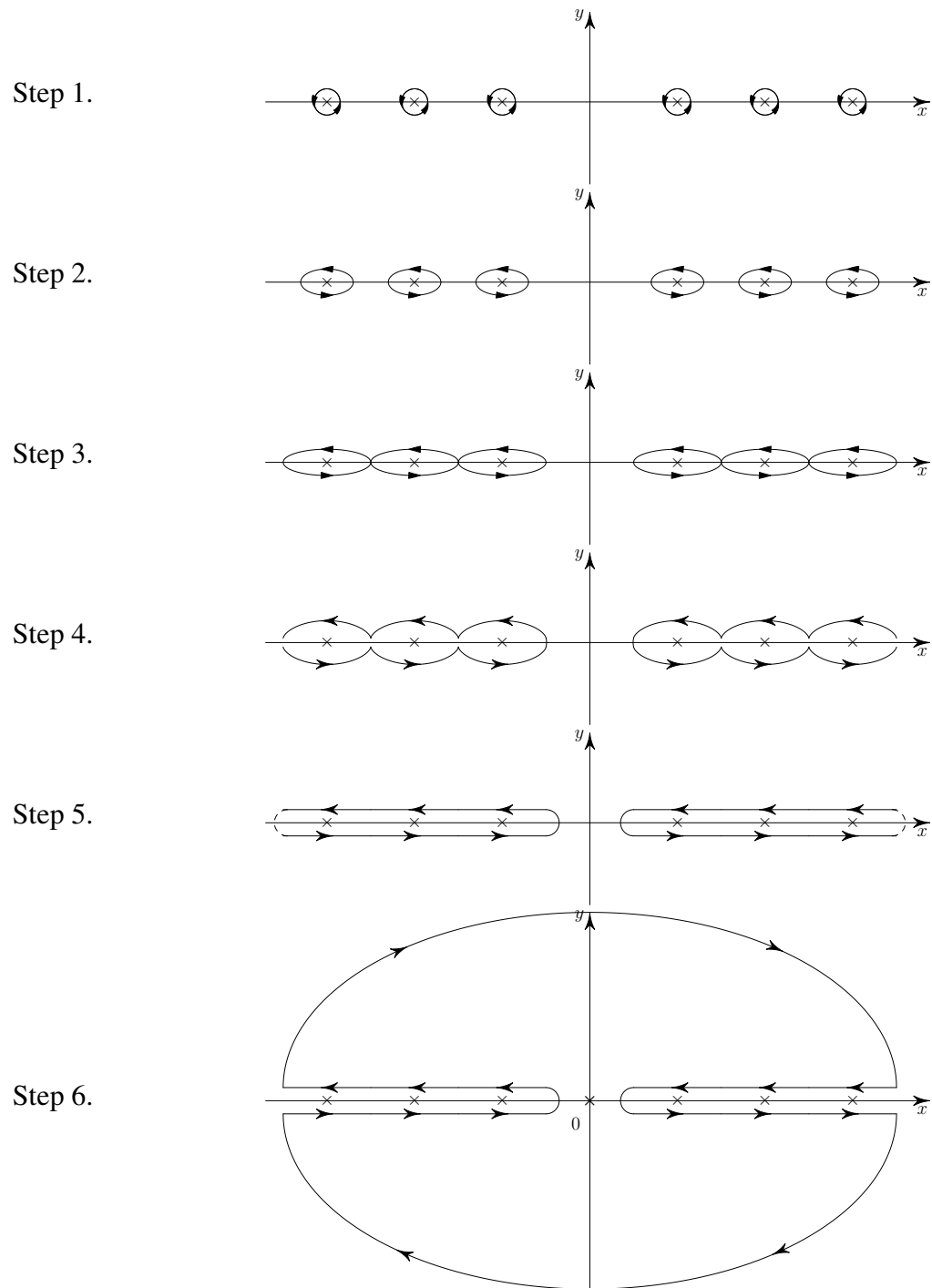


Figure 2: Transformation of the integration contour for Eq. (21)

2. The sum in Eq. (22) almost has the form of Eq. (2). Therefore we can use the developed technique summarized in Eqs. (7), (3), with some adjustments. Here

$$f(z) = \frac{1}{z^2}. \quad (23)$$

$f(z)$ has second order pole at $z = 0$.

$$F(z) = \frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}. \quad (24)$$

$z = 0$ is the third order pole of $F(z)$. Therefore,

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2} = -\text{Res} \left(\frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = 0 \right). \quad (25)$$

Using a computer algebra system, we find that the Laurent series for $F(z)$ is as following:

$$\frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} - \frac{2\pi^6 z^3}{945} - \frac{\pi^8 z^5}{4725} + \dots \quad (26)$$

Thus

$$\text{Res} \left(\frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = 0 \right) = -\frac{\pi^2}{3}. \quad (27)$$

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \text{Res} \left(\frac{\pi}{z^2} \frac{\cos(\pi z)}{\sin(\pi z)}, z = 0 \right) = \frac{\pi^2}{6} \quad (28)$$

Example 3. Let's evaluate the following sum:

$$S(a, b) = \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} \frac{1}{n^2 + b^2}, \quad (29)$$

where a, b are real and $a \neq b$, using the result Eq. (20):

$$S(a) = \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \frac{\cosh(\pi a)}{\sinh(\pi a)}.$$

Since

$$\frac{1}{n^2 + a^2} \frac{1}{n^2 + b^2} = \frac{1}{b^2 - a^2} \left(\frac{1}{n^2 + a^2} - \frac{1}{n^2 + b^2} \right), \quad (30)$$

$$S(a, b) = \frac{1}{b^2 - a^2} (S(a) - S(b)). \quad (31)$$

Therefore,

$$S(a, b) = \frac{1}{2a^2 b^2} + \frac{1}{b^2 - a^2} \left(\frac{\pi}{2a} \frac{\cosh(\pi a)}{\sinh(\pi a)} - \frac{\pi}{2b} \frac{\cosh(\pi b)}{\sinh(\pi b)} \right). \quad (32)$$

Example 4. Ramanujan's formula[4] Show that:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700} \quad (33)$$

Since the terms in the sum are even functions of n ,

$$S = \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\coth(\pi n)}{n^7}. \quad (34)$$

Here

$$f(z) = \frac{\coth(\pi z)}{z^7} \quad (35)$$

and

$$F(z) = \pi \frac{\coth(\pi z)}{z^7} \frac{\cos(\pi z)}{\sin(\pi z)}. \quad (36)$$

$F(z)$ has poles at n and in , for $n = 0, \pm 1, \pm 2, \dots$. The residue at real $z = n \neq 0$ is

$$\text{Res}(F(z), z = n) = \frac{\coth(\pi n)}{n^7}. \quad (37)$$

The residue at imaginary $z = in \neq 0$ is

$$\text{Res}(F(z), z = in) = \frac{1}{(in)^7} \frac{\cos(i\pi n)}{\sin(i\pi n)} = \frac{1}{n^7} \frac{\cosh(\pi n)}{\sinh(\pi n)} = \frac{\coth(\pi n)}{n^7} = \text{Res}(F(z), z = n). \quad (38)$$

Thus,

$$S = \sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^7} = -\frac{1}{4} \text{Res}(F(z), z = 0). \quad (39)$$

The Laurent series for $F(z)$ is as following:

$$F(z) = \frac{1}{\pi z^9} - \frac{7\pi^3}{45z^5} - \frac{19\pi^7}{14175z} - \frac{2906\pi^{11}z^3}{212837625} + \dots \quad (40)$$

Therefore, $z = 0$ is a 9th order pole of $F(z)$, with the residue

$$\text{Res}(F(z), z = 0) = -\frac{19\pi^7}{14175}. \quad (41)$$

$$S = -\frac{1}{4} \text{Res}(F(z), z = 0) = \frac{19\pi^7}{56700}. \quad (42)$$

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Let's consider another class of sums:

$$S = \sum_{n=-\infty}^{\infty} (-1)^n f(n). \quad (43)$$

The following function,

$$F(z) = \frac{\pi f(z)}{\sin(\pi z)}. \quad (44)$$

This function has simple poles at $z = n$, $n = 0, \pm 1, \pm 2, \dots$ with residues

$$\begin{aligned} \text{Res}(F(z), z = n) &= \lim_{z \rightarrow n} \frac{\pi f(z)}{\frac{d}{dz} \sin(\pi z)} = \lim_{z \rightarrow n} \frac{\pi f(z)}{\pi \cos(\pi z)} = \frac{f(n)}{\cos(\pi n)} = \frac{f(n)}{(-1)^n} = (-1)^n f(n). \\ \text{Res}(F(z), z = n) &= (-1)^n f(n). \end{aligned} \quad (45)$$

The poles of $f(z)$, $z = z_j$, are also poles of $F(z)$ that are different from the poles at $z = n$.

Using the same reasoning that we used in Sec. 1, arrive at the expression:

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = - \sum_j \text{Res}(F(z), z = z_j), \quad (46)$$

where the summation on the right is over the poles of $f(z)$ and $F(z)$ is given by Eq. (44)

Example 5. Let's evaluate the following sum:

$$S(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2}, \quad (47)$$

where a is a real positive parameter.

1. Convert the summation over the integer n from $[0, \infty)$ to $(-\infty, \infty)$. Since the terms in the sum are even functions of n ,

$$S(a) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} + \frac{1}{2a^2}, \quad (48)$$

where the last term on the right of Eq. (48) takes into account that the term with $n = 0$ appears in the summation only once.

2. The sum in Eq. (48) has the form of Eq. (45). Therefore we can use the developed technique summarized in Eqs. (46), (44). Here

$$f(z) = \frac{1}{z^2 + a^2}. \quad (49)$$

$f(z)$ has two simple poles at

$$z_{1,2} = \pm ia. \quad (50)$$

$$F(z) = \frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}. \quad (51)$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = -\text{Res} \left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia \right) - \text{Res} \left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia \right). \quad (52)$$

The first term on the right,

$$\text{Res} \left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia \right) = \lim_{z \rightarrow ia} \left[\frac{(z - ia)}{(z + ia)(z - ia)} \frac{\pi}{\sin(\pi z)} \right] = \frac{1}{2ia} \frac{\pi}{\sin(i\pi a)}. \quad (53)$$

Using Eq. (16)

$$\text{Res} \left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia \right) = \frac{1}{2ia} \frac{\pi}{i \sinh(\pi a)} = -\frac{1}{2a} \frac{\pi}{\sinh(\pi a)}. \quad (54)$$

Similarly,

$$\text{Res} \left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia \right) = \frac{1}{2ia} \frac{\pi}{i \sinh(\pi a)} = -\frac{1}{2a} \frac{\pi}{\sinh(\pi a)}. \quad (55)$$

Therefore, from Eqs. (52), (54), and (55)

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a} \frac{1}{\sinh(\pi a)}, \quad (56)$$

and, finally, from Eq. (48),

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \frac{1}{\sinh(\pi a)}. \quad (57)$$

Example 6.

$$S(x) = \frac{\sin x}{1 + a^2} - \frac{2 \sin 2x}{4 + a^2} + \frac{3 \sin 3x}{9 + a^2} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2 + a^2} \quad (58)$$

$$S(x) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2 + a^2} \quad (59)$$

$$f(z) = \frac{z \sin zx}{z^2 + a^2} \quad (60)$$

$$F(z) = \frac{z \sin zx}{z^2 + a^2} \frac{\pi}{\sin(\pi z)} \quad (61)$$

$$S(x) = \frac{1}{2} \text{Res} \left(\frac{z \sin zx}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia \right) + \frac{1}{2} \text{Res} \left(\frac{z \sin zx}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia \right) \quad (62)$$

$$\text{Res} \left(\frac{z \sin zx}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia \right) = \frac{ia \sin iax}{2ia} \frac{\pi}{\sin(\pi ia)} = \frac{\pi \sinh ax}{2 \sinh a\pi} \quad (63)$$

$$\text{Res} \left(\frac{z \sin(zx)}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia \right) = \frac{(-ia) \sin(-iax)}{(-2ia)} \frac{\pi}{\sin(-\pi ia)} = \frac{\pi \sinh(ax)}{2 \sinh(a\pi)} \quad (64)$$

$$S(x) = \frac{\pi \sinh(ax)}{2 \sinh(a\pi)} \quad (65)$$

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So far our examples dealt with the functions having isolated singularities in the complex plane. Let's consider an example involving integrands with branch points.

Example 7.

$$S = \sum_{n=-\infty}^{\infty} \log \left(\frac{b^2 + n^2}{a^2 + n^2} \right), \quad (66)$$

where a and b are real and positive; we assume that $b > a$. The function

$$\begin{aligned} f(z) &= \log \left(\frac{b^2 + z^2}{a^2 + z^2} \right) = \log \left(\frac{(z - ib)(z + ib)}{(z - ia)(z + ia)} \right) \\ &= \log(z - ib) + \log(z + ib) - \log(z - ia) - \log(z + ia) \end{aligned} \quad (67)$$

has branch points at $z = \pm ia, \pm ib$. The function $f(z)$ is analytic in the complex plane with two branch cuts – from ia to ib and from $-ia$ to $-ib$. The deformed contour circulates around the branch cuts. The real part of the integrals cancels as we integrate forward from ia to ib and back. The imaginary part is double its value in forward direction.

$$\pi \int_{ia}^{ib} \frac{\cos(\pi z)}{\sin(\pi z)} dz = \int_{ia}^{ib} \frac{d(\sin(\pi z))}{\sin(\pi z)} = \log(\sin(\pi z)) \Big|_{ia}^{ib} = \log \left(\frac{\sinh(\pi b)}{\sinh(\pi a)} \right). \quad (68)$$

References

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