Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form. Write your name on the problems page and enclose it together with your solutions. Enclose printouts of computer algebra sessions, where relevant.

Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	Total
Points:	20	25	20	35	100
Score:					

Jordan's lemma

1. (20 points) Consider a resistance R and inductance L connected in series with a voltage $V(t)$ (see Fig. 1). Suppose $V(t)$ is a voltage impulse at time $t = 0$, that is, a very strong pulse lasting

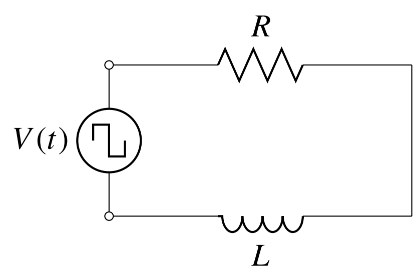

Figure 1: Series R-L circuit.

for a very short time around $t = 0$. As we shall see later in the course, we can write

$$V(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, \mathrm{d}\omega,$$

where A is the area under the curve $V(t)$. The current due to a voltage $v_\omega(t) = e^{i\omega t}$ is

$$i_\omega(t) = \frac{e^{i\omega t}}{R + i\omega}.$$

Thus the current due to our voltage pulse is

$$I(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{R + i\omega L}\, \mathrm{d}\omega. \qquad (*)$$

Evaluate the integral Eq. (*). Consider separately the cases $t < 0$ and $t > 0$, i.e. consider separately the times before and after the pulse.

Birthday paradox

2. In probability theory, the birthday paradox concerns the probability, $p(n)$, that, in a group of n randomly chosen people, at least one pair has the same birthday. The paradox, which is described below in more details, is a good illustrations how intuition can lead one astray in probability theory.

   For simplicity we assume that there are $M = 365$ possible birthdays in a year that are equally likely.

   The probability $p(n)$ is 1 when the number of people exceeds 365 (since there are only 365 possible birthdays). It seems plausible that the probability that a pair of people in a group has the same birthday is $\approx \frac{1}{2}$ when the number of people is close to $\frac{M}{2} \approx 182$.

   Your task in this assignment is to analyze the analytic expression for $p(n)$ and to check how close the "educated" guess above is to the correct answer. The analytic expression for the probability is derived for you as following:

   The problem is to compute the probability, $p(n)$ that in a group of n people, at least two have the same birthday. However, it is simpler to calculate $p_1(n)$, the probability that no two people in the group have the same birthday. Since the two events are mutually exclusive, $p(n) = 1 - p_1(n)$.

   Let's consider the probability of an elementary event, $P(i)$, that the person No. i in the group is not sharing his/her birthday with any of the previous $i - 1$ people. For Person 1, there are no previously analyzed people. Therefore, the probability, $P(1)$ is 1. The probability of 1 can also be written as

   $$P(1) = 365/365 = 1 - 0/365,$$

   for reasons that will become clear below.

   For Person 2, the only previously analyzed people is Person 1. The probability, $P(2)$, that Person 2 has a different birthday than Person 1 is

   $$P(2) = 364/365 = 1 - 1/365.$$

   Indeed, if Person 2 was born on any of the other 364 days of the year, Persons 1 and 2 do not share the same birthday.

   Similarly, if Person 3 is born on any of the 363 days of the year other than the birthdays of Persons 1 and 2, Person 3 will not share their birthday. This makes the probability $P(3)$

   $$P(3) = 363/365 = 1 - 2/365.$$

   This analysis continues until the last person in the group, Person n is reached, whose probability of not sharing his/her birthday with $(n - 1)$ people analyzed before, $P(n)$, is

   $$P(n) = \frac{365 - n + 1}{365} = 1 - \frac{n - 1}{365}.$$

   The probability that no two people in the group have the same birthday, $p_1(n)$, is equal to the product of individual probabilities:

   $$p_1(n) = P(1) \cdot P(2) \cdot P(3) \ldots P(n - 1) \cdot P(n)$$

or

$$p_1(n) = 1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \cdots \times \left(1 - \frac{n-1}{365}\right) = \prod_{k=0}^{n-1} \left(1 - \frac{k}{365}\right).$$

Alternatively,

$$p_1(n) = \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n} = \frac{365!}{365^n (365 - n)!}.$$

(a) (20 points) Evaluate the following product:

$$p_1(n, M) = \prod_{k=0}^{n-1} \left(1 - \frac{k}{M}\right),$$

which is the probability that not a single pair in group of size n shares the same birthday, assuming that there are M equally probable birthdates in a year

Hint: use Euler-Maclaurin summation formula for $\log p_1(n, M)$; keep only the integral term; to simplify the answer, assume that $n/M \ll 1$.

(b) (5 points) **Plot** a graph of the function $p_1(n)$. Use the graph to estimate how big a group of martians (martian year in martian days is $M = 668$) should be so that there is a 50-50 chance of two martians having the same birthday. Compare your answer with the "educated guess" from the earlier description in this problem.

The method of residues

3. (20 points) A charge e moving along a straight line undergoes simple harmonic motion with frequency ω and amplitude a. Since the charge is moving with an acceleration, it emits electromagnetic radiation. The angular distribution of the instantaneous (time-dependent) intensity of the radiation is

$$I(\theta,t) = \kappa \frac{\sin^2\theta \, \cos^2 \omega t'}{(1 + \beta \, \cos\theta \, \sin\omega t')^5} = \kappa \frac{\sin^2\theta \, \sin^2 \omega t}{(1 + \beta \, \cos\theta \, \cos\omega t)^5},$$

where θ is the angle between the direction of the oscillations and the direction of the emission, κ is an irrelevant for us constant,

$$\beta = \frac{\omega a}{c}, \quad 0 \le \beta < 1$$

is the ratio of the maximal speed of the charge to the speed of light. The time dependence of $I(\theta,t)$ is represented in the animation at http://www.phys.uconn.edu/phys2400/downloads/animation.gif for $\beta = 0.4$

Find the angular distribution of the averaged intensity of the emission,

$$I(\theta) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} I(\theta,t) \, \mathrm{d}t.$$

Use the contour integration. Sketch the integration contour in the complex plane and the singularitie(s) of the integrand. Use a computer algebra system to calculate the relevant residue(s).

Laplace methods

4. (a) (15 points) Use the Laplace method to solve the following boundary value problem in the form of a definite integral:

$$x\frac{d^5 y}{dx^5} + 4y = 0, \qquad y(0) = 1, \qquad y(\infty) = 0, \qquad 0 \le x < \infty.$$

Hint: Use the negative real axis as the complex integration contour. Verify that the contour fulfills the requirements of the Laplace method.

(b) (20 points) Use the "other" Laplace method to find the leading term in the asymptotics of your solution for $x \ll 1$. **Plot** your approximation on the same graph as the numerically evaluated integral solution for $2 \le x \le 10$. Use the logarithmic scale for y axis.