Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form. Write your name on the problems page and enclose it together with your solutions.

Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	5	Total
Points:	10	10	10	10	20	60
Score:						

Gamma function. Leibniz's rule.

1. (10 points) Find the value of x that minimizes the value of the following integral

$$I(x) = \int_x^{x+1} \ln \Gamma(u)\, \mathrm{d}u.$$

Complex algebra.

2. (10 points) Explain what is wrong with the following "proof":

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{(-1)}\,\sqrt{(-1)} = i \cdot i = -1$$

Derivative of complex functions. Cauchy-Riemann equations.

3. (a) (5 points) Can the following two real functions, $u(x, y)$ and $v(x, y)$, be the real and imaginary parts of an analytic function $f(z)$? Explain.

$$u(x, y) = x \cos y, \quad v(x, y) = y \sin x.$$

(b) (5 points) Can the following function, $u(x, y)$, be a real part of an analytic function inside a unit circle in the complex plane? Explain.

$$u(x, y) = x^2 + y^2$$

Hint: do not try to find the imaginary part, $v(x, y)$.

4. (10 points) Use Cauchy-Riemann equations to find the analytic function $f(z)$, $z = x + iy$, such that its real part is as following:

$$\mathrm{Re}\, f(z) = u(x, y) = x + y,$$

and

$$f(0) = 0.$$

Express the result for $f(z)$ as a **function of z only.**

Cauchy integral theorem.

5. (20 points) Show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha\phi\,(\cos\phi)^{i}\,\alpha - 1\mathrm{d}\phi = 2^{\alpha} B(\alpha, \alpha),$$

(where $B(x, y)$ is Euler's Beta function) by considering the integral

$$J_0 = \oint_C [z(1-z)]^{\alpha-1}\,\mathrm{d}z,$$

where the contour C (see Fig. 1) a union of the segment of the real axis $0 \le x \le 1$ and the semicircular arc

$$z = \frac{1}{2} + \frac{i}{2}e^{i\phi}, \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}.$$

Consider the case $\alpha \ge 1$.

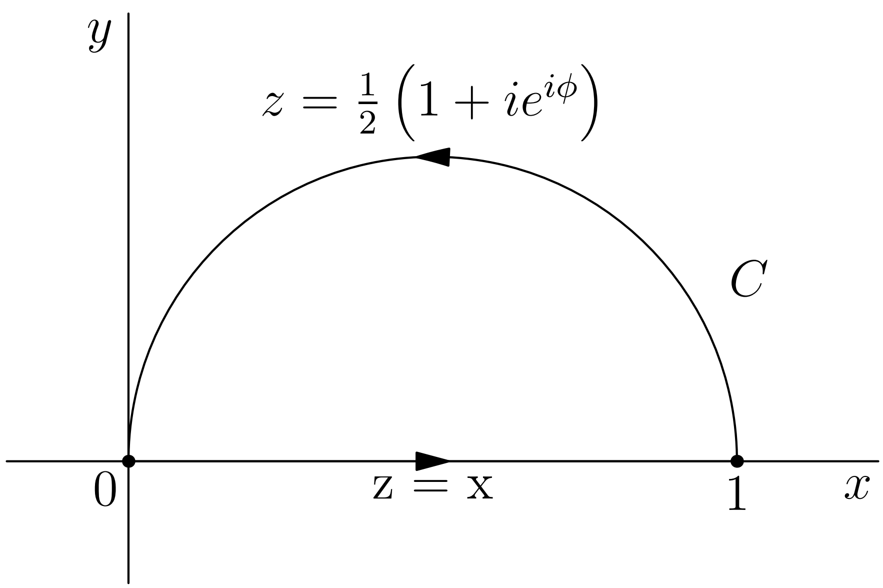

Figure 1: Integration contour for Problem 5.