Laplace method for integrals

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Laplace's method is a general technique for obtaining the asymptotic behavior as $x \to \infty$ of integrals in which the large parameter x appears in the exponent:

$$I(x) = \int_a^b f(t) e^{x\phi(t)} \mathrm{d}t. \tag{1}$$

Here $f(t)$ and $\phi(t)$ are real continuous functions. Integrals of this form are called Laplace integrals. Laplace's method rests on an idea involved in many techniques of asymptotic analysis of integrals: if the real continuous function $\phi(t)$ has its maximum on the interval $a < t < b$ at $t = t_0$ and if $f(t_0) \neq 0$, then it is only the immediate neighborhood of $t = t_0$ that contributes to the asymptotic expansion of $I(x)$ for large x.

Example 1. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^{4} e^{-\lambda x^2} \log\left(1 + x^2\right) \mathrm{d}x. \tag{2}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important,

$$\log\left(1 + x^2\right) \sim x^2. \tag{3}$$

$$I(\lambda) \sim \int_{-3}^{4} e^{-\lambda x^2} x^2 \,\mathrm{d}x \sim \int_{-\infty}^{\infty} e^{-\lambda x^2} x^2 \,\mathrm{d}x = 2 \int_{0}^{\infty} e^{-\lambda x^2} x^2 \,\mathrm{d}x. \tag{4}$$

New integration variable,

$$u = \lambda x^2 \quad \longrightarrow \quad x^2 = \frac{u}{\lambda} \quad \longrightarrow \quad x = \frac{1}{\sqrt{\lambda}} u^{\frac{1}{2}} \quad \longrightarrow \quad \mathrm{d}x = \frac{1}{2\sqrt{\lambda}} u^{-\frac{1}{2}} \mathrm{d}u. \tag{5}$$

$$I(\lambda) \sim \frac{1}{\lambda^{\frac{3}{2}}} \int_{0}^{\infty} e^{-u} u^{\frac{1}{2}} \,\mathrm{d}u = \frac{1}{\lambda^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\lambda^{\frac{3}{2}}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{2\lambda^{\frac{3}{2}}}} \tag{6}$$

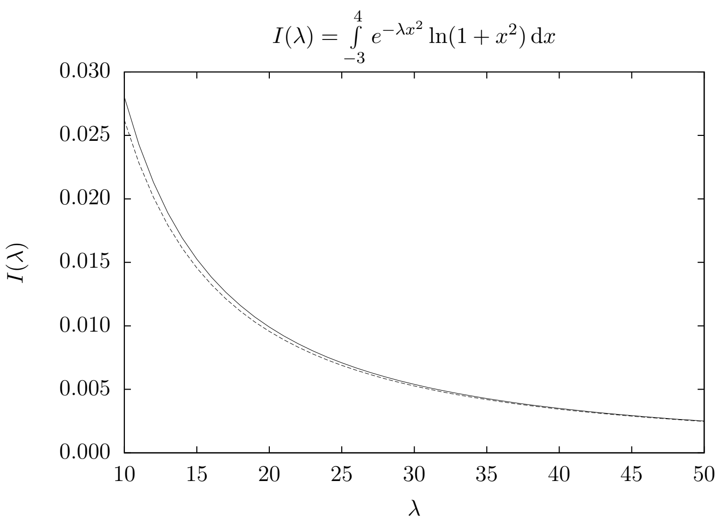

Figure 1: Asymptotics Eq. (6) (solid line) compared to the numerically evaluated Eq. (2) (dashed line) for $10 \le \lambda \le 100$.

Example 2. Find the leading term of the asymptotics of the following integral for $n \gg 1$:

$$I(n) = \int_{-1}^{1} (\cos x)^n \, \mathrm{d}x, \tag{7}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important,

$$\cos x \sim 1 - \frac{x^2}{2} \sim e^{-\frac{x^2}{2}}. \tag{8}$$

$$I(n) = \int_{-1}^{1} \left(e^{-\frac{x^2}{2}}\right)^n \mathrm{d}x \sim \int_{-\infty}^{\infty} e^{-\frac{nx^2}{2}} \, \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{n}{2}}x\right)^2} \mathrm{d}x = \tag{9}$$

$$= \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{n}{2}}x\right)^2} \mathrm{d}\left(\sqrt{\frac{n}{2}}x\right) = \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-u^2} \, \mathrm{d}u = \tag{10}$$

$$= \boxed{\sqrt{\frac{2\pi}{n}}}. \tag{11}$$

Example 3. Find the leading term of the asymptotics of the solution of the following differential equation, $y(x)$, for $x \to \infty$:

$$x y''' + 2y = 0, \quad y(0) = 1, \quad y(\infty) = 0, \quad y'(\infty) = 0. \tag{12}$$

The solution of Eq. (12) is as following:

$$y(x) = \int_{0}^{\infty} e^{-t - \frac{x}{\sqrt{t}}} \, \mathrm{d}t. \tag{13}$$

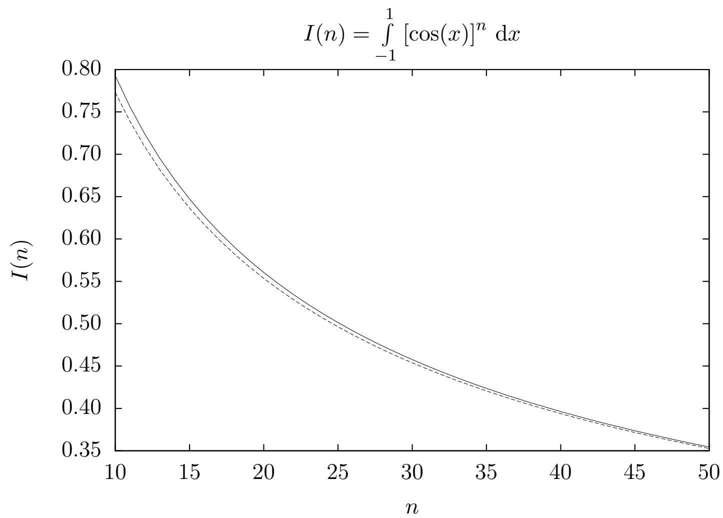

Figure 2: Asymptotics Eq. (11) (solid line) compared to the numerically evaluated integral (7) (dashed line) for $10 \le n \le 100$.

To verify this, let's differentiate $y(x)$ three times with respect to x and integrate by parts:

$$y''' = -\int_0^\infty e^{-t}\, e^{-\frac{x}{\sqrt{t}}}\, \frac{1}{t^{\frac{3}{2}}}\, \mathrm{d}t, \tag{14}$$

$$e^{-\frac{x}{\sqrt{t}}}\, \frac{1}{t^{\frac{3}{2}}}\, \mathrm{d}t = \frac{2}{x}\, \mathrm{d}\left(e^{-\frac{x}{\sqrt{t}}}\right), \tag{15}$$

$$y''' = -\frac{2}{x}\int_0^\infty e^{-t}\, \mathrm{d}\left(e^{-\frac{x}{\sqrt{t}}}\right) = -\frac{2}{x}\left[e^{-t-\frac{x}{\sqrt{t}}}\Big|_0^\infty - \int_0^\infty e^{-\frac{x}{\sqrt{t}}}\, \mathrm{d}\left(e^{-t}\right)\right] = -\frac{2\, y(x)}{x}. \tag{16}$$

Thus indeed, $x y''' + 2y = 0$.

The function in the exponent in Eq. (13),

$$f(t) = -t - \frac{x}{\sqrt{t}}, \tag{17}$$

has its maximum at $t = t_0$ which depends upon x:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0, \quad \longrightarrow \quad -1 + \frac{x}{2t^{\frac{3}{2}}} = 0 \quad \longrightarrow \quad t_0 = 2^{-\frac{2}{3}} x^{\frac{2}{3}}. \tag{18}$$

To make the maximum independent of x, let's introduce a new integration variable, s,

$$s = \frac{t}{x^{\frac{2}{3}}}, \quad \longrightarrow \quad t = x^{\frac{2}{3}} s, \quad \longrightarrow \quad \mathrm{d}t = x^{\frac{2}{3}}\, \mathrm{d}s, \quad \sqrt{t} = x^{\frac{1}{3}}\sqrt{s}, \quad t + \frac{x}{\sqrt{t}} = x^{\frac{2}{3}}\left(s + \frac{1}{\sqrt{s}}\right) \tag{19}$$

$$y(x) = x^{\frac{2}{3}} \int_0^\infty e^{-x^{\frac{2}{3}}\left(s + \frac{1}{\sqrt{s}}\right)}\, \mathrm{d}s. \tag{20}$$

To simplify equations below, let's introduce the notation

$$\lambda = x^{\frac{2}{3}}. \tag{21}$$

$$y(x) = \lambda \int_0^\infty e^{-\lambda\left(s+\frac{1}{\sqrt{s}}\right)}\, \mathrm{d}s \tag{22}$$

$$f(s) = s + \frac{1}{\sqrt{s}} \quad \rightarrow \quad f'(s) = 1 - \frac{1}{2\, s^{\frac{3}{2}}} \quad \rightarrow \quad s_0 = 2^{-\frac{2}{3}}, \quad f(s_0) = 2^{-\frac{2}{3}} + 2^{\frac{1}{3}} = 3 \cdot 2^{-\frac{2}{3}}. \tag{23}$$

$$f'' = \frac{3}{4\, s^{\frac{5}{2}}} \quad \rightarrow \quad f''(s_0) = \frac{3}{4}\left(2^{-\frac{2}{3}}\right)^{-\frac{5}{2}} = 3 \cdot 2^{-\frac{1}{3}}. \tag{24}$$

In the vicinity of s_0

$$f(s) = f(s_0) + \frac{1}{2} f''(s_0)\,(s - s_0)^2 = 3 \cdot 2^{-\frac{2}{3}} + 3 \cdot 2^{-\frac{4}{3}}(s - s_0)^2. \tag{25}$$

$$y \sim \lambda \int_0^\infty e^{-\lambda\left(3\cdot 2^{-\frac{2}{3}} + 3\cdot 2^{-\frac{4}{3}}(s-s_0)^2\right)}\, \mathrm{d}s = \lambda\, e^{-3\lambda 2^{-\frac{2}{3}}} \int_{\infty}^{\infty} e^{-3\lambda 2^{-\frac{4}{3}}(s-s_0)^2}\, \mathrm{d}s = \lambda\, e^{-3\lambda 2^{-\frac{2}{3}}} \sqrt{\pi} \left(3\lambda 2^{-\frac{4}{3}}\right)^{-\frac{1}{2}}. \tag{26}$$

Simplifying, and restoring the original notations, arriving at the following expression:

$$y(x) \sim \boxed{\pi^{\frac{1}{2}} 3^{-\frac{1}{2}} 2^{\frac{2}{3}}\, x^{\frac{1}{3}}\, e^{-3\left(\frac{x}{2}\right)^{\frac{2}{3}}}}. \tag{27}$$

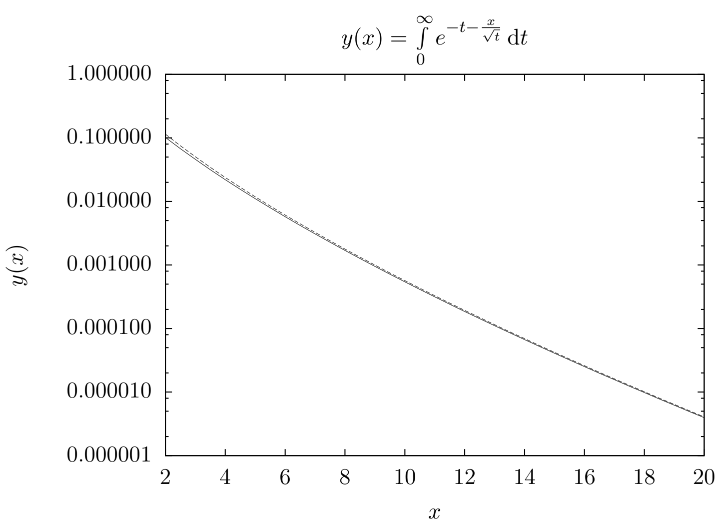

Figure 3: Asymptotics Eq. (27) (solid line) compared to the numerically evaluated integral (13) (dashed line) for $2 \le x \le 20$. Notice the logarithmic scale on y axis.

Example 4. Find the leading term of the asymptotics of gamma function, $\Gamma(x)$, for $x \to \infty$:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, \mathrm{d}t = \int_0^\infty e^{-t + x\log t} \frac{1}{t}\, \mathrm{d}t \tag{28}$$

The function in the exponent in Eq. (28),

$$f(t) = -t + x\log t, \tag{29}$$

has its maximum at $t = t_0$ which depends upon x:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0, \quad \longrightarrow \quad -1 + \frac{x}{t} = 0 \quad \longrightarrow \quad t_0 = x. \tag{30}$$

To make the maximum independent of x, let's introduce a new integration variable, s,

$$s = \frac{t}{x}, \quad \longrightarrow \quad t = xs, \quad \longrightarrow \quad \mathrm{d}t = x\,\mathrm{d}s, \quad \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}s}{s}, \tag{31}$$

$$f(t) = -t + x\,\log t = -xs + x\,\log s + x\,\log x. \tag{32}$$

$$\Gamma(x) = e^{x\,\log x} \int_0^\infty e^{-x(s - \log s)} \frac{1}{s}\, \mathrm{d}s. \tag{33}$$

Let's apply the Laplace's method to the integral in Eq. (33):

$$f(s) = s - \log s, \quad \frac{\mathrm{d}f}{\mathrm{d}s} = 1 - \frac{1}{s}. \tag{34}$$

$$\frac{\mathrm{d}f}{\mathrm{d}s} = 0, \quad \longrightarrow \quad s_0 = 1. \tag{35}$$

$$f(s_0) = 1, \quad \frac{\mathrm{d}^2 f}{\mathrm{d}s^2} = \frac{1}{s^2}, \quad \longrightarrow \quad \frac{\mathrm{d}^2 f}{\mathrm{d}s^2}(s_0) = 1. \tag{36}$$

$$f(s) \approx f(s_0) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}(s_0)(s - s_0)^2 = 1 + \frac{1}{2}(s-1)^2. \tag{37}$$

$$\int_0^\infty e^{-xf(s)} \frac{1}{s}\, \mathrm{d}s \sim \int_0^\infty e^{-x\left(1 + \frac{1}{2}(s-1)^2\right)} \frac{1}{s_0}\, \mathrm{d}s \sim e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}x(s-1)^2}\, \mathrm{d}s = e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}xs^2}\, \mathrm{d}s. \tag{38}$$

The last integral is a Gaussian one:

$$\int_{-\infty}^\infty e^{-\frac{1}{2}xs^2}\, \mathrm{d}s = \sqrt{\frac{2\pi}{x}}, \tag{39}$$

therefore

$$\int_0^\infty e^{-x(s - \log s)} \frac{1}{s}\, \mathrm{d}s \sim e^{-x}\sqrt{\frac{2\pi}{x}}. \tag{40}$$

Finally, combining Eq. (33) and Eq. (40)

$$\Gamma(x) \sim e^{x\,\log x} e^{-x} \sqrt{\frac{2\pi}{x}} = \boxed{\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x} \tag{41}$$

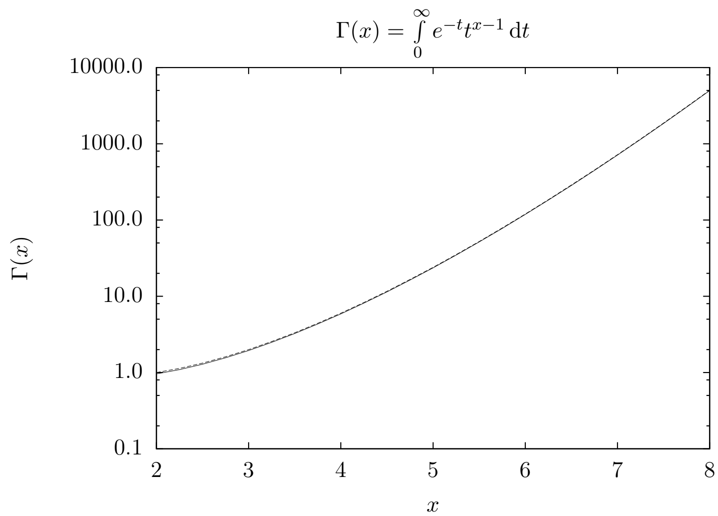

Figure 4: Asymptotics Eq. (41) (solid line) compared to the numerically evaluated integral (28) (dashed line) for $2 \le x \le 8$. Notice the logarithmic scale on y axis.

Example 5. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 e^{-\lambda \sin(x)} \frac{\cos^3(x^2)}{\log(2 + x^2)}\, dx. \tag{42}$$

The maximum of the function in the exponent, $-\sin x$ is outside the integration range, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin x \sim x$.

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x} \frac{\cos^3(0)}{\log(2 + 0)}\, dx = \frac{1}{\log 2} \int_0^\infty e^{-\lambda x}\, dx = \boxed{\frac{1}{\lambda \log(2)}}. \tag{43}$$

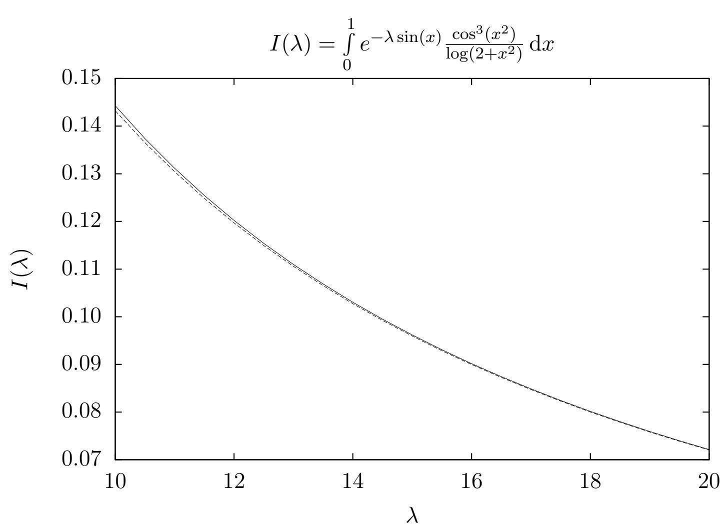

Figure 5: Asymptotics Eq. (43) (solid line) compared to the numerically evaluated integral (42) (dashed line) for $10 \le \lambda \le 20$.

Example 6. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 e^{-\lambda \sin^6(x)} \frac{\cos^3(x^2)}{\log(2+x^2)}\, dx. \tag{44}$$

The maximum of the function in the exponent, $-\sin^6 x$ is outside the integration range, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin^6 x \sim x^6$.

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x^6} \frac{\cos^3(0)}{\log(2+0)}\, dx = \frac{1}{\log 2} \int_0^\infty e^{-\lambda x^6}\, dx. \tag{45}$$

To evaluate the last integral, let's introduce a new integration variable, $u = \lambda x^6$:

$$x^6 = \frac{u}{\lambda} \quad \longrightarrow \quad x = \frac{u^{\frac{1}{6}}}{\lambda^{\frac{1}{6}}} \quad \longrightarrow \quad dx = \frac{u^{\frac{1}{6}-1}}{6\lambda^{\frac{1}{6}}}\, du. \tag{46}$$

$$I(\lambda) \sim \frac{1}{6\lambda^{\frac{1}{6}}} \int_0^\infty e^{-u} u^{\frac{1}{6}-1}\, du = \boxed{\frac{\Gamma(\frac{1}{6})}{6\lambda^{\frac{1}{6}} \log(2)}}. \tag{47}$$

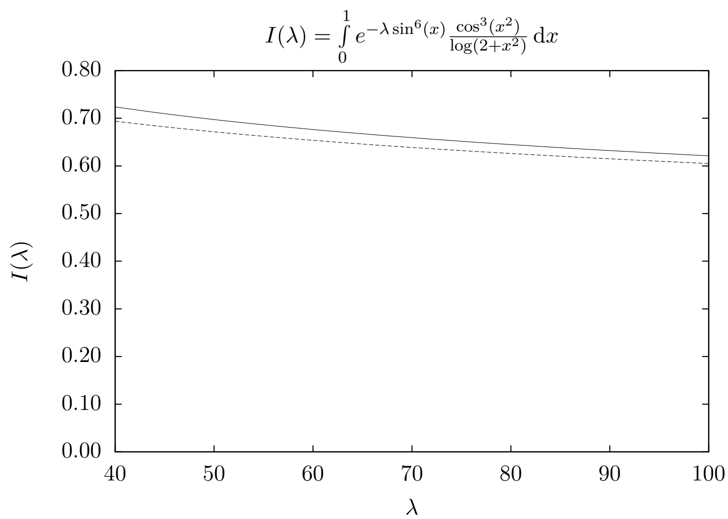

Figure 6: Asymptotics Eq. (47) (solid line) compared to the numerically evaluated integral (46) (dashed line) for $40 \le \lambda \le 100$.

References

[1] Lorella M. Jones. *An introduction to mathematical methods of physics*. Benjamin Cummings, 1979.

[2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.