Name: ____________________________

Date: ____________________________

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points: | 10 | 25 | 15 | 10 | 10 | 10 | 80 |
| Score: | | | | | | | |

**Isolated singularities in the complex plane.**

1. Consider the following function of the complex variable $z$

$$f(z) = \frac{1}{(z-i)(z-5)^2} \sin\left(\frac{1}{z}\right).$$

   (a) (2 points) Characterize the singularity at $z = i$.

   - ☐ Simple pole
   - ☐ Second order pole
   - ☐ Essential singularity
   - ☐ The function is analytic at $z = i$

   (b) (4 points) Characterize the singularity at $z = 5$.

   - ☐ Simple pole
   - ☐ Second order pole
   - ☐ Essential singularity
   - ☐ The function is analytic at $z = 5$

   (c) (4 points) Characterize the singularity at $z = 0$.

   - ☐ Simple pole
   - ☐ Second order pole
   - ☐ Essential singularity
   - ☐ The function is analytic at $z = 0$

**Sumation of series**

2. (25 points) Evaluate the following two sums reducing them to contour integral(s):

$$S(a) = \sum_{n=-\infty}^{\infty} \frac{1}{n^4 + a^4}, \qquad T(a) = \sum_{n=-\infty}^{\infty} \frac{n^2}{n^4 + a^4},$$

where $a$ is a positive parameter.

Sketch the integration contour(s). Indicate the position(s) of the pole(s) of the integrand. Compare your answer with the result produced by a computer algebra system.

Hint: (a) obtain the sum for $T(a) + ia^2 S(a)$ and separate the real and the imaginary parts of your answer. (b) Use a computer algebra system to separate the parts.

## Operators

Let $\hat{D} \equiv \frac{\mathrm{d}}{\mathrm{d}x}$.

3. (15 points) Evaluate the following expression:

$$\left(\frac{1}{\left(1+\hat{D}\right)^2}\right) x^2.$$

4. (10 points) Recall that shift (aka translation) operator $\hat{T} \equiv e^{\hat{D}}$ has the following property:

$$\hat{T} f(x) = f(x+1).$$

Which of the expressions below (there may be more than one) are equal to $f(x+6)$?

- ☐ $6\hat{T} f(x)$
- ☐ $3\hat{T}^2 f(x)$
- ☐ $2\hat{T}^3 f(x)$
- ☐ $\hat{T}^6 f(x)$
- ☐ $\left(\hat{T}^2\right)^3 f(x)$
- ☐ $\left(\hat{T}^3\right)^2 f(x)$
- ☐ All of the above
- ☐ None of the above

## Euler-MacLaurin summation formula

Recall that

$$\begin{aligned}\sum_{n=m}^{N+m} f(n) &= \sum_{n=0}^{N} f(m+n) = \int_m^{m+N} f(x)\mathrm{d}x + \frac{1}{2}\left(f(N+m) + f(m)\right) + \\ &\quad + \frac{1}{12}\left(f'(N+m) - f'(m)\right) - \frac{1}{720}\left(f'''(N+m) - f'''(m)\right) + \ldots\end{aligned}$$

5. (10 points) Evaluate the following sum using the first two terms in Euler-MacLaurin formula.

$$S = \sum_{n=2}^{\infty} \frac{1}{n \log(n)^{\frac{3}{2}}}.$$

6. (10 points) Evaluate the following product:

$$\alpha = \prod_{n=1}^{\infty} n^{1/n^2}.$$

Hint: Use Euler-MacLaurin summation formula to evaluate $\log(\alpha)$. Consider only the integral term of the expression.