The Gamma function meets physics[1]

In the following example we see how the Gamma functions appears in a problem involving the one-dimensional motion of a point mass under the influence of a potential force with the following potential:

$$V(x) = \alpha x^p, \tag{1}$$

where x is the coordinate of the particle.

The case $p = 2$ corresponds to the motion of a harmonic oscillator, $p = -1$ describes the motion of a charge in a coulomb field, $p = -3$ describes dipole-dipole interaction, etc. Our analysis is not restricted to particular values of p. We assume that the force on the mass is directed toward the origin, $x = 0$, i.e. that $\alpha p > 0$.

Specifically, a mass m is held at rest at a point with the coordinate $x = x_0$. We assume that $x_0 > 0$. The mass is allowed to move at time $t = 0$. At what time, T, does the mass arrive at the origin?

We consider separately the cases of positive and negative p.

1 $V(x) = \alpha x^p$ for $p > 0$

Conservation of energy gives the velocity of the mass in terms of its position:

$$\frac{m\dot{x}^2}{2} + \alpha x^p = \alpha x_0^p, \tag{2}$$

where the term on the right is the initial energy of the point mass; $\dot{x}$ denotes the time derivative of x, i.e. the velocity of the mass.

$$\dot{x}^2 = \frac{2\alpha}{m}\left(x_0^p - x^p\right), \tag{3}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\frac{2\alpha x_0^p}{m}}\left[1 - \left(\frac{x}{x_0}\right)^p\right]^{\frac{1}{2}}. \tag{4}$$

The sign on the right is chosen such that the velocity is directed toward the attracting origin.

Separating variables in the ordinary differential equation Eq. (4),

$$-\left[1 - \left(\frac{x}{x_0}\right)^p\right]^{-\frac{1}{2}}\mathrm{d}x = \sqrt{\frac{2\alpha x_0^p}{m}}\,\mathrm{d}t. \tag{5}$$

Integrating both sides of Eq. (5), on the left with respect to x, from x_0 to 0, and on the right with respect to t, from 0 to T, obtain:

$$-\int_{x_0}^{0}\left[1 - \left(\frac{x}{x_0}\right)^p\right]^{-\frac{1}{2}}\mathrm{d}x = \sqrt{\frac{2\alpha x_0^p}{m}}\,T. \tag{6}$$

[1] The title borrowed from P. Nahin, *Inside Interesting Integrals*, Springer, 2015.

In the integral on the left we swap the limits of integration to compensate the minus sign. Next we introduce the new integration variable,

$$u = \left(\frac{x}{x_0}\right)^p, \quad 0 \le u \le 1, \quad x = x_0 u^{\frac{1}{p}}, \quad \mathrm{d}x = \frac{x_0}{p} u^{\frac{1}{p}-1} \mathrm{d}u. \tag{7}$$

$$T_p = \frac{1}{p}\sqrt{\frac{m}{2\alpha x_0^{p-2}}} \int_0^1 u^{\frac{1}{p}-1} (1-u)^{-\frac{1}{2}} \,\mathrm{d}u = \frac{1}{p}\sqrt{\frac{m}{2\alpha x_0^{p-2}}}\, B\left(\frac{1}{p}, \frac{1}{2}\right), \tag{8}$$

where $B(\ldots)$ is the Beta function, or

$$\boxed{T_p = \frac{1}{p}\sqrt{\frac{m}{2\alpha x_0^{p-2}}}\, \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} = \frac{1}{p}\sqrt{\frac{m\pi}{2\alpha x_0^{p-2}}}\, \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}}. \tag{9}$$

As a quick check of the result Eq. (9) consider the motion of a harmonic oscillator, $p = 2$:

$$T_2 = \frac{1}{2}\sqrt{\frac{m\pi}{2\alpha}}\, \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\pi}{2}\sqrt{\frac{m}{2\alpha}}, \tag{10}$$

which is indeed one quarter of the period of oscillations and doesn't depend upon x_0.

2 $V(x) = \alpha x^p$ for $p < 0$

Conservation of energy gives the velocity of the mass in terms of its position:

$$\frac{m\dot{x}^2}{2} + \alpha x^p = \alpha x_0^p, \tag{11}$$

where the term on the right is the initial energy of the point mass; $\dot{x}$ denotes the time derivative of x, i.e. the velocity of the mass.

$$\dot{x}^2 = \frac{2|\alpha|}{m}\left(x^p - x_0^p\right) = \frac{2|\alpha|}{m}\left(\frac{1}{x^{|p|}} - \frac{1}{x_0^{|p|}}\right), \tag{12}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\frac{2|\alpha|}{m}}\, x^{-|p|/2}\left[1 - \left(\frac{x}{x_0}\right)^{|p|}\right]^{\frac{1}{2}}. \tag{13}$$

The sign on the right is chosen such that the velocity is directed toward the attracting origin.

Separating variables in the ordinary differential equation Eq. (13),

$$-\, x^{|p|/2}\left[1 - \left(\frac{x}{x_0}\right)^{|p|}\right]^{-\frac{1}{2}} \mathrm{d}x = \sqrt{\frac{2|\alpha|}{m}}\, \mathrm{d}t. \tag{14}$$

Integrating both sides of Eq. (5), on the left with respect to x, from x_0 to 0, and on the right with respect to t, from 0 to T, obtain:

$$-\int_{x_0}^{0} x^{|p|/2}\left[1-\left(\frac{x}{x_0}\right)^{|p|}\right]^{-\frac{1}{2}} \mathrm{d}x = \sqrt{\frac{2|\alpha|}{m}}\,T. \tag{15}$$

In the integral on the left we swap the limits of integration to compensate the minus sign. Next we introduce the new integration variable,

$$u = \left(\frac{x}{x_0}\right)^{|p|}, \quad 0 \le u \le 1, \quad x = x_0 u^{\frac{1}{|p|}}, \quad \mathrm{d}x = \frac{x_0}{|p|} u^{\frac{1}{|p|}-1}\mathrm{d}u. \tag{16}$$

$$T_p = \frac{1}{|p|}\sqrt{\frac{m\,x_0^{|p|+2}}{2|\alpha|}} \int_0^1 u^{\frac{1}{|p|}+\frac{1}{2}-1}\,(1-u)^{\frac{1}{2}-1}\,\mathrm{d}u = \frac{1}{|p|}\sqrt{\frac{m\,x_0^{|p|+2}}{2|\alpha|}}\,B\left(\frac{1}{|p|}+\frac{1}{2},\frac{1}{2}\right), \tag{17}$$

where $B(\ldots)$ is the Beta function, or

$$\boxed{T_p = \frac{1}{|p|}\sqrt{\frac{m}{2|\alpha|x_0^{p-2}}}\,\frac{\Gamma\left(\frac{1}{|p|}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{|p|}+1\right)} = \frac{1}{|p|}\sqrt{\frac{m\pi}{2|\alpha|x_0^{p-2}}}\,\frac{\Gamma\left(\frac{1}{|p|}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{|p|}+1\right)}}. \tag{18}$$