

Frullani's integral

Consider the following integrals:

$$I_1(a, b) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx, \quad (1)$$

$$I_2(a, b) = \int_0^{\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx, \quad (2)$$

$$I_3(a, b) = \int_0^{\infty} \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} dx. \quad (3)$$

They are all of the form

$$I(a, b) = \int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx, \quad (4)$$

which is called Frullani's integral, after the Italian mathematician Giuliano Frullani (1795–1834). Integral Eq. (4) can be evaluated for arbitrary $f(x)$ (assuming that the integral exists). The formula for the value of integral (4), our Eq. (18), was first published by Cauchy in 1823. About 1829 Frullani published the same formula and mentioned that he had communicated it to Plana (Italian astronomer and mathematician, 1781–1864) in 1821.

To reproduce the Cauchy's result, we do as following:

1. Reduce the number of effective parameters in the integral:

let's introduce a new integration variable, t .

$$t = bx, \quad 0 \leq t < \infty, \quad ax = \frac{a}{b}t, \quad \frac{dx}{x} = \frac{d(bx)}{bx} = \frac{dt}{t}. \quad (5)$$

$$I(a, b) = \int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = \int_0^{\infty} \frac{f\left(\frac{a}{b}t\right) - f(t)}{t} dt \quad (6)$$

The integral depends upon the single parameter – the ratio of a and b .

$$\alpha = \frac{a}{b} \quad (7)$$

$$I(a, b) \equiv I(\alpha) = \int_0^{\infty} \frac{f(\alpha t) - f(t)}{t} dt \quad (8)$$

2. Let's differentiate Eq. (8) with respect to α thus obtaining a differential equation for $I(\alpha)$:

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha} \int_0^{\infty} \frac{f(\alpha t) - f(t)}{t} dt = \int_0^{\infty} \left[\frac{d}{d\alpha} f(\alpha t) \right] \frac{dt}{t} \quad (9)$$

Using the chain rule

$$\frac{d}{d\alpha} f(\alpha t) = \frac{df(\alpha t)}{d(\alpha t)} \frac{d(\alpha t)}{d\alpha} = \frac{df(\alpha t)}{d(\alpha t)} t. \quad (10)$$

Therefore,

$$\frac{dI}{d\alpha} = \int_0^{\infty} \frac{df(\alpha t)}{d(\alpha t)} dt = \frac{1}{\alpha} \int_0^{\infty} \frac{df(\alpha t)}{d(\alpha t)} d(\alpha t) \quad (11)$$

$$= \frac{1}{\alpha} \int_0^{\infty} \frac{df(u)}{du} du = \frac{f(\infty) - f(0)}{\alpha}, \quad (12)$$

where in Eq. (12) we introduced a new independent variable, $u = \alpha t$, and used the fundamental theorem of calculus to evaluate the integral.

$$\frac{dI}{d\alpha} = \frac{f(\infty) - f(0)}{\alpha} \quad (13)$$

3. Let's integrate the differential equation and determine the integration constant.

Eq. (13) is a first order ordinary differential equation that can be solved separating variables.

$$dI = \frac{d\alpha}{\alpha} [f(\infty) - f(0)] \quad (14)$$

Integrating both sides of Eq. (14),

$$I(\alpha) = [f(\infty) - f(0)] \ln \alpha + C, \quad (15)$$

where C is an integration constant. C can be determined by noticing that the value of integral (4) is zero when $a = b$. (Equivalently, integral Eq. (8) is 0 when $\alpha = 1$.) Therefore

$$I(1) = 0, \quad (16)$$

thus

$$C = 0. \quad (17)$$

Finally, combining Eqs. (15), (17), and (8), we arrive to the following expression:

$$\boxed{\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = [f(\infty) - f(0)] \ln \frac{a}{b}}, \quad (18)$$

This result assumes that $f(x)$ is such that both $f(0)$ and $f(\infty)$ exist.

Eq. (18) is the main result. Let's apply it to evaluate the integrals Eqs. (1)–(3).

$$1. I_1 = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx:$$

$$f(x) = e^{-x}, f(0) = 1, f(\infty) = 0, \text{ thus } I_1 = -\ln \frac{a}{b} = \boxed{\ln \frac{b}{a}}$$

$$2. I_2 = \int_0^{\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int_0^{\infty} \frac{e^{-(\sqrt{a}x)^2} - e^{-(\sqrt{b}x)^2}}{x} dx:$$

$$f(x) = e^{-x^2}, f(0) = 1, f(\infty) = 0, \text{ thus } I_2 = -\ln \frac{\sqrt{a}}{\sqrt{b}} = \boxed{\frac{1}{2} \ln \frac{b}{a}}$$

$$3. I_3 = \int_0^{\infty} \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} dx:$$

$$f(x) = e^{-x} \cos x, f(0) = 1, f(\infty) = 0, \text{ thus } I_3 = \boxed{\ln \frac{b}{a}}$$

Let's consider the following integral:

$$I_4 = \int_0^{\infty} \sin px \sin qx \frac{dx}{x}. \quad (19)$$

It doesn't exactly look like Frullani's integral, however recall that

$$\sin px \sin qx = \frac{1}{2} (\cos(p-q)x - \cos(p+q)x). \quad (20)$$

This,

$$I_4 = \frac{1}{2} \int_0^{\infty} \frac{\cos(p-q)x - \cos(p+q)x}{x} dx. \quad (21)$$

Integral (21) does look like a frullani integral, with $f(x) = \cos(x)$. Unfortunately we are still in trouble, since $\cos(\infty)$ doesn't exist. A common trick to resolve the problem is to consider

$$f(x) = e^{-tx} \cos(x), \quad (22)$$

where t is a positive parameter, and consider the result in the limit $t \rightarrow +0$:

$$\lim_{t \rightarrow +0} f(0) = 1, \quad \lim_{t \rightarrow +0} f(\infty) = 0, \quad I_4 = \lim_{t \rightarrow +0} \left\{ [f(\infty) - f(0)] \log \frac{p-q}{p+q} \right\} = \log \frac{p+q}{p-q}. \quad (23)$$