Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form. Write your name on the problems page and enclose it together with your solutions. Enclose printouts of computer algebra sessions, where relevant.

Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

Nonlinear differential equations. Method of averaging

1. (25 points) Find the amplitude and the period of the *limit cycle* of the Rayleigh oscillator for small positive values of the "dissipation" ε.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left(\frac{1}{3}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 - \frac{\mathrm{d}x}{\mathrm{d}t}\right) + x = 0,$$

   To compare your result and the numerical solution of the Rayleigh equation for $\varepsilon = 0.1$ and the initial conditions $x(0) = 1.$, $\dot{x}(0) = 0$, and plot two graphs on the same figure.

2. (25 points) A longcase clock, also tall-case clock, floor clock, or grandfather clock, is a tall, freestanding, weight-driven pendulum clock with the pendulum held inside the tower. The English clockmaker William Clement is credited with the development of this form in 1670. Until the early 20th century, pendulum clocks were the world's most accurate timekeeping technology, and longcase clocks, due to their superior accuracy, served as time standards for households and businesses.

   The oscillations of the pendulum are described by the equation

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0,$$

   where ϕ is the angle between the pendulum and the vertical axis, l is the length of the pendulum, and g is the acceleration of gravity.

   The pendulum of a longcase clock clock swings to a maximum angle of $5^{\angle}$ from the vertical. How many seconds does the clock gain or lose each day (1 day = 86400 sec) if the clock is adjusted to keep perfect time when the angular swing is infinitesimaly small?

   Hints:

   1. Since the swing angle in the problem are small, simplify the equation for the pendulum while still keeping it nonlinear.

2. The relative loss or gain of the clock is proportional to relative change in frequencies of oscillations

3. Do not forget to conver angles to radians.

General analytical technique. Properties of Beta and Gamma functions

3. Find the expression containing only elementary functions for the following product of two Gamma functions:
$$I(x) = \Gamma(x)\Gamma(1-x).$$
Your result is going to be valid for arbitrary x, however for your derivation you may assume that x is real and $0 < x < 1$.

   The suggested way to solve the problem is as following:

   (a) Use the relation between Gamma and Beta functions and find the relation between $I(x)$ and $B(x, 1-x)$. Next, use the definition of beta function and express $I(x)$ as an integral:
$$I(x) = \int_0^1 \mathrm{d}t \ldots$$

   (b) (5 points) Use the change of the integration variable
$$u = \frac{t}{1-t} \quad \longleftrightarrow \quad t = \frac{u}{1+u}$$
to change the integration limits to $[0, \infty)$:
$$I(x) = \int_0^\infty \mathrm{d}u \ldots$$

   (c) (5 points) Make another change of integration variable
$$u = e^v$$
to change the integration limits to $(-\infty, \infty)$:
$$I(x) = \int_{-\infty}^{\infty} \mathrm{d}v \ldots$$

   (d) (10 points) If you did all the transformation above correctly, you should arrive to the following expression:
$$I(x) = \int_{-\infty}^{\infty} \frac{e^{xv}}{1+e^v} \,\mathrm{d}v.$$
Evaluate the integral using the method of residues. The suggested integration contour is presented in Fig. 1.

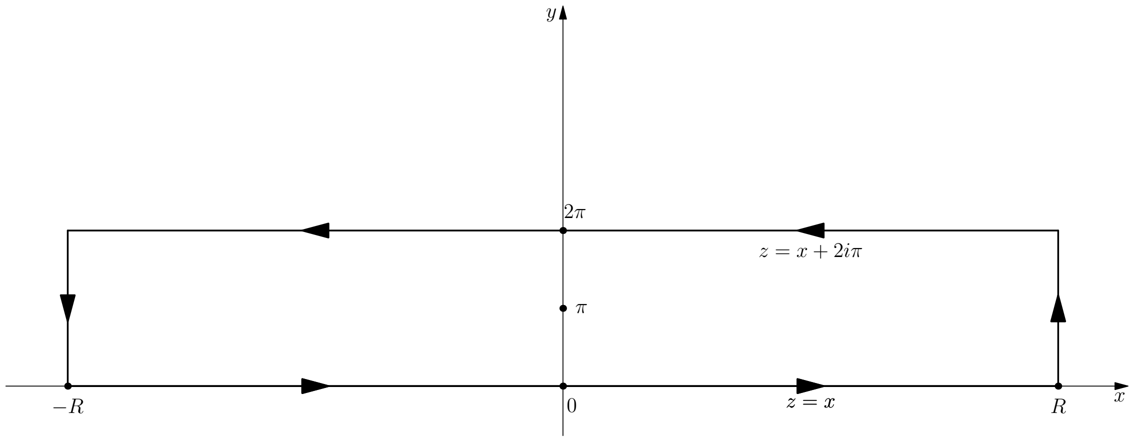

Figure 1: Integration contour for Problem 3(d).

(e) (5 points) Collect all your results together. Verify that your answer is correct for $x = \frac{1}{2}$ (recall that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$).

Singular perturbation methods

4. (25 points) To be discussed in class:

   Consider the following boundary value problem:

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + e^y = 0, \qquad y(0) = y(1) = 0.$$

   Determine the position of a boundary layer, find the leading terms if the inner and outer solutions, and construct the uniform approximation for the solution.

   On the same graph plot the numerical solution of the problem and your approximation for $\varepsilon = 0.1$.