Euler's formula

1 Introduction

In complex analysis Euler's formula establishes the relation between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number θ:

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1}$$

where e is the base of the natural logarithm, i is the imaginary unit, and $\cos$ and $\sin$ are the trigonometric functions, with the argument θ given in radians. The formula Eq. (1) was published by Euler in 1748, obtained by comparing the series expansions of the exponential and trigonometric expressions.

2 Derivation

Consider

$$z(\theta) = \cos\theta + i\sin\theta, \tag{2}$$

where θ is a real parameter.

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\sin\theta + i\cos\theta = i\left(\cos\theta + i\sin\theta\right) = iz. \tag{3}$$

$$\frac{\mathrm{d}z}{z} = i\mathrm{d}\theta. \tag{4}$$

$$\ln z = i\theta + C', \tag{5}$$

where C' is an integration constant.

$$z(t) = Ce^{i\theta}, \tag{6}$$

where $C = e^{C'}$ is another constant. From Eq. (2),

$$z(0) = 1. \tag{7}$$

From Eq. (6)

$$z(0) = C. \tag{8}$$

Therefore, $C = 1$ and

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}. \tag{9}$$

3 Geometric interpretation and the complex plain

A point in the complex plane can be represented by a complex number written in cartesian coordinates. Euler's formula provides a means of conversion between cartesian coordinates and polar coordinates. The polar form simplifies the mathematics when used in multiplication, division, or powers of complex numbers.

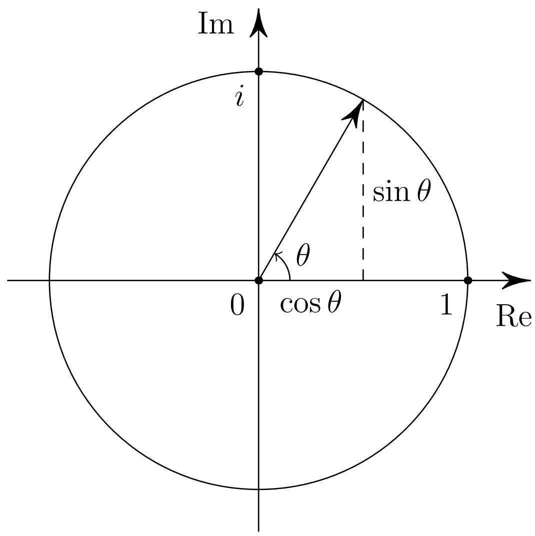

Figure 1: Euler's formula illustrated in the complex plane.