Induced emf in a circular loop[1]

A circular wire loop of resistance R and radius a has its center at a distance $d_0 > a$ from a long straight wire. The wire is in the plane of the loop. (See Fig. 1.) The current in the long wire is changing, $I = I(t)$. What is the current in the loop?

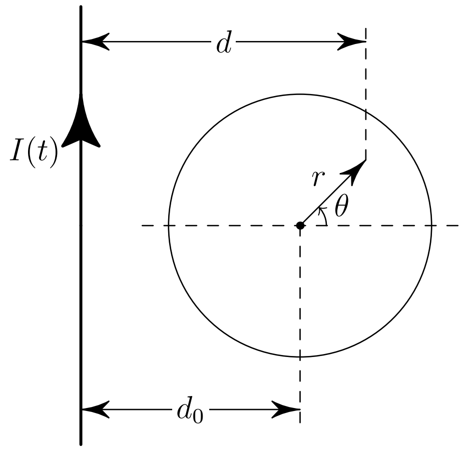

Figure 1: A long straight wire carrying current $I(t)$ and a circular wire loop whose center is distance d_0 from the wire.

The emf induced in the loop has the magnitude

$$\varepsilon = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\int B_n \mathrm{d}A\right|, \tag{1}$$

where Φ is the magnetic flux through the loop,

$$\Phi = \int B_n \mathrm{d}A, \tag{2}$$

B_n is the component of the magnetic field perpenducular to the plane of the loop; the integration is over the area of the loop. The magnetic field produced by the wire at a radial distance d from the wire,

$$B_n = \frac{\mu_0 I}{2\pi d}. \tag{3}$$

At a point inside the loop,

$$d = d_0 + r\cos\theta \tag{4}$$

and

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta, \tag{5}$$

so that the flux through the loop is

$$\Phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\int_0^a \frac{r\mathrm{d}r\mathrm{d}\theta}{d_0 + r\cos\theta} = \frac{\mu_0 I}{2\pi}\int_0^a \phi(r)\, r\,\mathrm{d}r. \tag{6}$$

[1]The physics problem in this note is “borrowed” from: S. Lea, *Mathematics for Physicists*, Brooks Cole, 2003

The integral over θ,

$$\phi(r) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{d_0 + r\cos\theta}. \tag{7}$$

Change the integration variable:

$$z = e^{i\theta}, \quad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{8}$$

In the z-plane the integration contour is the unit circle $|z| = 1$.

$$\phi(r) = -\frac{2i}{r} \oint_{|z|=1} \frac{\mathrm{d}z}{z^2 + 2\frac{d_0}{r}z + 1} \tag{9}$$

The poles of the integrand are given by the roots of the quadratic polynomial in the denominator:

$$z^2 + 2\frac{d_0}{r}z + 1 = 0. \tag{10}$$

$$z_{in,out} = -\frac{d_0}{r} \pm \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}. \tag{11}$$

Since the loop is not crossing the wire, $d_0 > a \geq r$, i.e. $\frac{d_0}{r} > 1$. Therefore both z_{in} and z_{out} are real. Since $z_{in}z_{out}$ = 1, only one root, the one with the smaller absolute value, is withing the integration contour:

$$z_{in} = -\frac{d_0}{r} + \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}. \tag{12}$$

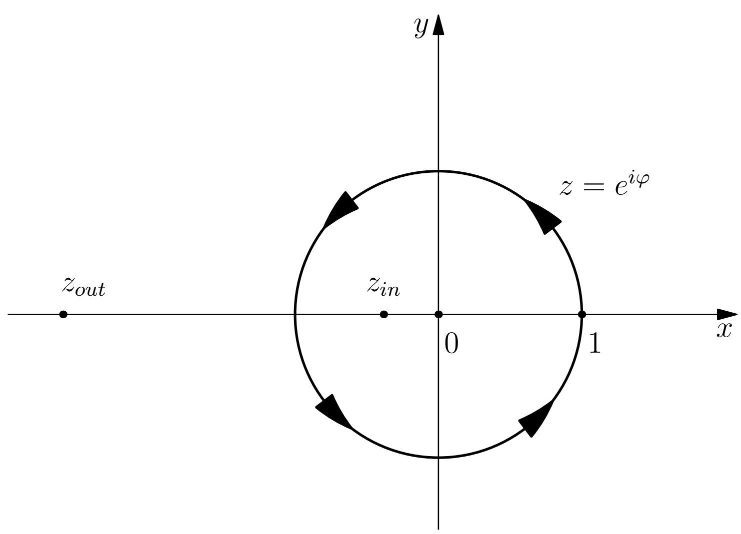

Figure 2: Integration contour for Eq. (9). The poles of the integrand, z_{in} and z_{out}, are shown for $\frac{d_0}{r} = \frac{3}{2}$.

$$\begin{aligned} \phi(r) &= 2\pi i\left(-\frac{2i}{r}\right)\mathrm{Res}\left(\frac{1}{(z-z_{in})(z-z_{out})}, z = z_{in}\right) && (13)\\ &= \frac{4\pi}{r}\frac{1}{z_{in}-z_{out}} = \frac{2\pi}{r}\frac{1}{\sqrt{\left(\frac{d_0}{r}\right)^2 - 1}} = \frac{2\pi}{\sqrt{d_0^2 - r^2}}. && (14)\end{aligned}$$

$$\Phi = \frac{\mu_0 I}{2\pi} \int_0^a \phi(r)\, r\, \mathrm{d}r = \mu_0 I \int_0^a \frac{r\, \mathrm{d}r}{\sqrt{d_0^2 - r^2}}. \tag{15}$$

Noting that

$$r\, \mathrm{d}r = \frac{1}{2} \mathrm{d}\left(r^2\right) = -\frac{1}{2} \mathrm{d}\left(d_0^2 - r^2\right) \tag{16}$$

and introducing new integration variable

$$u = d_0^2 - r^2, \quad d_0^2 - a^2 \leq u \leq d_0^2. \tag{17}$$

$$\Phi = \frac{\mu_0 I}{2} \int_{d_0^2 - a^2}^{d_0^2} \frac{\mathrm{d}u}{\sqrt{u}} = \mu_0 I\, \sqrt{u}\, \Big|_{d_0^2 - a^2}^{d_0^2} = \mu_0 I \left(d_0 - \sqrt{d_0^2 - a^2}\right). \tag{18}$$

We can check the result by looking at the limit $a \ll d_0$. We expect the ux to be approximately

$$\Phi \approx \frac{\mu_0 I}{2\pi d_0}\, \pi a^2 = \frac{\mu_0 I a^2}{2 d_0}. \tag{19}$$

Now if we expand the square root in Equation (18), we get

$$\Phi = \mu_0 I d_0 \left(1 - \sqrt{1 - \left(\frac{a}{d_0}\right)^2}\right) \approx \mu_0 I d_0 \left[1 - 1 + \frac{1}{2}\left(\frac{a^2}{d_0}\right)\right] = \frac{\mu_0 I a^2}{2 d_0} \tag{20}$$

as expected.

Finally, the current in the loop is

$$I = \frac{\varepsilon}{R} = \frac{\mu_0 \dot{I}}{R} \left(d_0 - \sqrt{d_0^2 - a^2}\right), \tag{21}$$

where $\dot{I} = \frac{\mathrm{d}I}{\mathrm{d}t}$.