Laplace method for integrals

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Laplace’s method is a general technique for obtaining the asymptotic behavior as $x \to \infty$ of integrals in which the large parameter $x$ appears in an exponential:

$$I(x) = \int_a^b f(t)e^{x\phi(t)} \, dt.$$  \hfill (1)

Here $f(t)$ and $\phi(t)$ are real continuous functions. Integrals of this form are called Laplace integrals. Laplace’s method rests on an idea involved in many techniques of asymptotic analysis of integrals: if the real continuous function $\phi(t)$ has its maximum on the interval $a < t < b$ at $t = t_0$ and if $f(t_0) \neq 0$, then it is only the immediate neighborhood of $t = t_0$ that contributes to the asymptotic expansion of $I(x)$ for large $x$.

**Example 1.** Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^4 e^{-\lambda x^2} \log (1 + x^2) \, dx.$$  \hfill (2)

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ are important,

$$\log (1 + x^2) \sim x^2.$$  \hfill (3)

$$I(\lambda) \sim \int_{-3}^4 e^{-\lambda x^2} x^2 \, dx \sim \int_{-\infty}^\infty e^{-\lambda x^2} x^2 \, dx = 2 \int_0^\infty e^{-\lambda x^2} x^2 \, dx.$$  \hfill (4)

New integration variable,

$$u = \lambda x^2 \quad \rightarrow \quad x^2 = \frac{u}{\lambda} \quad \rightarrow \quad x = \frac{1}{\sqrt{\lambda}} u^{\frac{1}{2}} \quad \rightarrow \quad dx = \frac{1}{2\sqrt{\lambda}} u^{-\frac{1}{2}} \, du.$$  \hfill (5)

$$I(\lambda) \sim \frac{1}{\lambda^{\frac{3}{2}}} \int_0^\infty e^{-u} u^{\frac{1}{2}} \, du = \frac{1}{\lambda^{\frac{3}{2}}} \Gamma \left( \frac{3}{2} \right) = \frac{1}{\lambda^{\frac{3}{2}}} \frac{1}{2^\frac{1}{2}} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2\lambda^{\frac{3}{2}}},$$  \hfill (6)
Figure 1: Asymptotics Eq. (6) (solid line) compared to the numerically evaluated Eq. (2) (dashed line) for $10 \leq \lambda \leq 100$.

Example 2. Find the leading term of the asymptotics of the following integral for $n \gg 1$:

$$I(n) = \int_{-1}^{1} (\cos x)^n \, dx,$$  \hspace{1cm} (7)

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{n}} \ll 1$, are important,

$$\cos x \sim 1 - \frac{x^2}{2} \sim e^{-\frac{x^2}{2}}.$$  \hspace{1cm} (8)

$$I(n) = \int_{-1}^{1} \left( e^{-\frac{x^2}{2}} \right)^n \, dx \sim \int_{-\infty}^{\infty} e^{-\frac{n x^2}{2}} \, dx = \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{n} x}{\sqrt{2}}\right)^2} \, dx =$$  \hspace{1cm} (9)

$$= \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{n} x}{\sqrt{2}}\right)^2} \, dx = \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-u^2} \, du =$$  \hspace{1cm} (10)

$$= \sqrt{\frac{2\pi}{n}}.$$  \hspace{1cm} (11)

Example 3. Find the leading term of the asymptotics of the solution of the following differential equation, $y(x)$, for $x \to \infty$:

$$x y''' + 2 y = 0, \quad y(0) = 1, \quad y(\infty) = 0, \quad y'(\infty) = 0.$$  \hspace{1cm} (12)

The solution of Eq. (12) is as following:

$$y(x) = \int_{0}^{\infty} e^{-t - \frac{t^2}{x}} \, dt.$$  \hspace{1cm} (13)
To verify this, let’s differentiate \( y(x) \) three times with respect to \( x \) and integrate by parts:

\[
y''' = -\int_0^\infty e^{-t} e^{-\frac{x}{\sqrt{t}}} \frac{1}{t^{\frac{3}{2}}} \, dt,
\]

\[
e^{-\frac{x}{\sqrt{t}}} \frac{1}{t^{\frac{3}{2}}} \, dt = \frac{2}{x} \left( e^{-\frac{x}{\sqrt{t}}} \right),
\]

\[
y''' = -\frac{2}{x} \int_0^\infty e^{-t} \left( e^{-\frac{x}{\sqrt{t}}} \right) = -\frac{2}{x} \left[ e^{-t-\frac{x}{\sqrt{t}}} \right]_0^\infty - \int_0^\infty e^{-t-\frac{x}{\sqrt{t}}} d\left( e^{-t} \right) = -\frac{2y(x)}{x}.
\]

Thus indeed, \( xy''' + 2y = 0 \).

The function in the exponent in Eq. (13),

\[
f(t) = -t - \frac{x}{\sqrt{t}},
\]

has its maximum at \( t = t_0 \) which depends upon \( x \):

\[
\frac{df}{dt} = 0, \quad \implies \quad -1 + \frac{x}{2t^\frac{3}{2}} = 0 \quad \implies \quad t_0 = 2^{-\frac{2}{3}}x^\frac{2}{3}.
\]

To make the maximum independent of \( x \), let’s introduce a new integration variable, \( s \),

\[
s = \frac{t}{x^\frac{2}{3}}, \quad \implies \quad t = x^\frac{2}{3}s, \quad \implies \quad dt = x^\frac{2}{3} \, ds, \quad \sqrt{t} = x^\frac{1}{3}\sqrt{s}, \quad t + \frac{x}{\sqrt{t}} = x^\frac{2}{3} \left( s + \frac{1}{\sqrt{s}} \right).
\]

\[
y(x) = x^\frac{2}{3} \int_0^\infty e^{-x^\frac{2}{3}\left( s + \frac{1}{\sqrt{s}} \right)} \, ds.
\]
To simplify equations below, let’s introduce the notation
\[ \lambda = x^{\frac{2}{3}}. \]
(21)

\[ y(x) = \lambda \int_0^\infty e^{-\lambda(s + \frac{1}{\sqrt{s}})} \, ds \]
(22)

\[ f(s) = s + \frac{1}{\sqrt{s}} \quad \Rightarrow \quad f'(s) = 1 - \frac{1}{2s^{\frac{3}{2}}} \quad \Rightarrow \quad s_0 = 2^{-\frac{2}{3}}, \quad f(s_0) = 2^{-\frac{2}{3}} + 2^{\frac{1}{4}} = 3 \cdot 2^{-\frac{2}{3}}. \]
(23)

\[ f'' = \frac{3}{4s^{\frac{5}{2}}} \quad \Rightarrow \quad f''(s_0) = \frac{3}{4} \left(2^{-\frac{2}{3}}\right)^{-\frac{5}{2}} = 3 \cdot 2^{-\frac{1}{2}}. \]
(24)

In the vicinity of \( s_0 \)

\[ f(s) = f(s_0) + \frac{1}{2} f''(s_0) (s - s_0)^2 = 3 \cdot 2^{-\frac{2}{3}} + 3 \cdot 2^{-\frac{4}{3}} (s - s_0)^2. \]
(25)

\[ y \sim \lambda \int_0^\infty e^{-\lambda(3 \cdot 2^{-\frac{2}{3}} + 3 \cdot 2^{-\frac{4}{3}}(s - s_0)^2)} \, ds = \lambda e^{-3\lambda 2^{-\frac{2}{3}}} \int_0^\infty e^{-3\lambda 2^{-\frac{4}{3}}(s - s_0)^2} \, ds = \lambda e^{-3\lambda 2^{-\frac{2}{3}}} \sqrt{\pi} \left(3\lambda 2^{-\frac{4}{3}}\right)^{-\frac{1}{2}}. \]
(26)

Simplifying, and restoring the original notations, arriving at the following expression:

\[ y(x) \sim \pi^{\frac{1}{3}} 3^{-\frac{1}{2}} 2^{\frac{2}{3}} x^{\frac{1}{2}} e^{-3\left(\frac{x}{2}\right)^{\frac{2}{3}}}. \]
(27)

**Figure 3:** Asymptotics Eq. (27) (solid line) compared to the numerically evaluated integral (13) (dashed line) for \( 2 \leq x \leq 20 \). Notice the logarithmic scale on the y axis.
Example 4. Find the leading term of the asymptotics of gamma function, $\Gamma(x)$, for $x \to \infty$:

$$\Gamma(x) = \int_0^\infty e^{-t} e^{x-1} \, dt = \int_0^\infty e^{-t+x \log t} \frac{1}{t} \, dt$$  \quad (28)

The function in the exponent in Eq. (28),

$$f(t) = -t + x \log t,$$  \quad (29)

has its maximum at $t = t_0$ which depends upon $x$:

$$\frac{df}{dt} = 0, \quad \implies \quad -1 + \frac{x}{t} = 0 \quad \implies \quad t_0 = x.$$  \quad (30)

To make the maximum independent of $x$, let’s introduce a new integration variable, $s$,

$$s = \frac{t}{x}, \quad \implies \quad t = x s, \quad \implies \quad \frac{dt}{x} = ds, \quad \frac{dt}{t} = \frac{ds}{s},$$  \quad (31)

$$f(t) = -t + x \log t = -xs + x \log s + x \log x.$$  \quad (32)

$$\Gamma(x) = e^{x \log x} \int_0^\infty e^{-x(s-\log s)} \frac{1}{s} \, ds.$$  \quad (33)

Let’s apply the Laplace’s method to the integral in Eq. (33):

$$f(s) = s - \log s, \quad \frac{df}{ds} = 1 - \frac{1}{s}.$$  \quad (34)

$$\frac{df}{ds} = 0, \quad \implies \quad s_0 = 1.$$  \quad (35)

$$f(s_0) = 1, \quad \frac{d^2 f}{ds^2} = \frac{1}{s^2}, \quad \implies \quad \frac{d^2 f}{ds^2}(s_0) = 1.$$  \quad (36)

$$f(s) \approx f(s_0) + \frac{1}{2} \frac{d^2 f}{ds^2}(s_0)(s-s_0)^2 = 1 + \frac{1}{2}(s-1)^2.$$  \quad (37)

$$\int_0^\infty e^{-xf(s)} \frac{1}{s} \, ds \sim \int_0^\infty e^{-x(1+\frac{1}{2}(s-1)^2)} \frac{1}{s_0} \, ds \sim e^{-x} \int_0^\infty e^{-\frac{1}{2}x(s-1)^2} \, ds = e^{-x} \int_0^\infty e^{-\frac{1}{2}xs^2} \, ds.$$  \quad (38)

The last integral is a Gaussian one:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}xs^2} \, ds = \sqrt{\frac{2\pi}{x}},$$  \quad (39)

therefore

$$\int_0^\infty e^{-x(s-\log s)} \frac{1}{s} \, ds \sim e^{-x} \sqrt{\frac{2\pi}{x}}.$$  \quad (40)

Finally, combining Eq. (33) and Eq. (40)

$$\Gamma(x) \sim e^{x \log x} e^{-x} \sqrt{\frac{2\pi}{x}} = \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x$$  \quad (41)
Figure 4: Asymptotics Eq. (41) (solid line) compared to the numerically evaluated integral (28) (dashed line) for $2 \leq x \leq 8$. Notice the logarithmic scale on $y$ axis.

Example 5. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 e^{-\lambda \sin(x)} \frac{\cos^3(x^2)}{\log(2 + x^2)} \, dx.$$  \hspace{1cm} (42)

The maximum of the function in the exponent, $- \sin x$ is outside the integration range, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin x \sim x$.

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x} \frac{\cos^3(0)}{\log(2 + 0)} \, dx = \frac{1}{\log 2} \int_0^\infty e^{-\lambda x} \, dx = \frac{1}{\lambda \log(2)}. \hspace{1cm} (43)$$

Figure 5: Asymptotics Eq. (43) (solid line) compared to the numerically evaluated integral (42) (dashed line) for $10 \leq \lambda \leq 20$. 
Example 6. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^{1} e^{-\lambda \sin^6(x)} \frac{\cos^3(x^2)}{\log(2 + x^2)} \, dx. \quad (44)$$

The maximum of the function in the exponent, $-\sin^6 x$ is outside the integration range, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin^6 x \sim x^6$.

$$I(\lambda) \sim \int_0^{\infty} e^{-\lambda x^6} \frac{\cos^3(0)}{\log(2 + 0)} \, dx = \frac{1}{\log 2} \int_0^{\infty} e^{-\lambda x^6} \, dx. \quad (45)$$

To evaluate the last integral, let’s introduce a new integration variable, $u = \lambda x^6$:

$$x^6 = \frac{u}{\lambda} \quad \rightarrow \quad x = \frac{u^{1/6}}{\lambda^{1/6}} \quad \rightarrow \quad dx = \frac{u^{1/6 - 1}}{6\lambda^{1/6}} \, du. \quad (46)$$

$$I(\lambda) \sim \frac{1}{6\lambda^{1/6}} \int_0^{\infty} e^{-u} u^{1/6 - 1} \, du = \frac{\Gamma\left(\frac{1}{6}\right)}{6\lambda^{1/6} \log(2)}. \quad (47)$$

Figure 6: Asymptotics Eq. (47) (solid line) compared to the numerically evaluated integral (46) (dashed line) for $40 \leq \lambda \leq 100$.

References
