AMAT-3132, Winter 2003.

Euler's derivation of the Euler-Maclaurin formula

1 Bernoulli numbers

In the present context Bernoulli numbers will appear as coefficients in the Euler-Maclaurin formula. They arise also, quite unexpectedly, in a number of other questions. See, for example, Lab 2A in this term's AMAT-2130
http://www.math.mun.ca/ m2130/winter2003/lab2A.ps

It is well known that the function $\dfrac{e^t-1}{t}$ has finite limit as $t \to 0$. Moreover, it can be expanded in Maclaurin's series

$$\frac{e^t-1}{t} = \frac{1}{t}\sum_{k=1}^{\infty}\frac{t^k}{k!} = 1 + \frac{t}{2} + \frac{t^2}{6} + \dots .$$

Bernoulli numbers B_k are, by definition, the values of the derivaives of the reciprocal function $\dfrac{t}{e^t-1}$ at $t=0$, that is we have Maclaurin's expansion

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \dots = 1 + \sum_{k=1}^{\infty}\frac{B_k}{k!} \tag{1}$$

with $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, etc. The Bernoulli numbers can be computed recurrently by equating (to zero) coefficients at powers of t in the identity

$$1 = \frac{e^x-1}{x}\cdot\frac{x}{e^x-1} = \left(1+\frac{x}{2}+\frac{x^2}{6}+\dots\right)\cdot\left(1+\frac{B_1}{1!}x+\frac{B_2}{2!}x+\dots\right).$$

B_1 is the only nonzero Bernoulli number with an odd subscript. Bernoulli numbers with even subscripts are all nonzero. Some first of them are

$$B_2 = \frac{1}{6}, \qquad B_4 = \frac{-1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = \frac{-1}{30}, \qquad B_{10} = \frac{5}{66},$$

In spite of an apparent smallness of the first terms of this sequence, the pattern is elusive; in fact, $B_{2n} \to \infty$ as $n \to \infty$ at a nearly factorial rate.

2 Operators $D = \frac{d}{dx}$ and T (shift)

If $f(x)$ is a "good" function (meaning that we can apply formulae of differential calculus we need without reservations), then the correspondence

$$f(x) \to f'(x)$$

can be regarded as the *operator of differentiation* $D = \frac{d}{dx}$ that acts on the function and transforms it into the derivative. Similarly, the correspondence

$$f(x) \to f(x+1)$$

can be regarded as the *shift operator* T that acts on the function and transforms it into a new function $Tf(x)$ defined pointwise as $Tf(x) := f(x+1)$.

The Taylor expansion

$$f(x+1) = f(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x)$$

can be interpreted as follows:

$$Tf(x) = f(x) + \left(\sum_{n=1}^{\infty} \frac{1}{n!} D^n \right) f(x)$$

If we associate the identity operator $f(x) \to f(x)$ with number 1 (saying, for instance, that $f(x)$ is being multiplied by the constant 1), then we can formally write the

$$T \;=\; 1 + \sum_{n=1}^{\infty} \frac{D^n}{n!} \;=\; e^D. \tag{2}$$

Of course, this is not a numerical identity, and its rigorous treatment is by far not straightforward. (Mathematical discipline that deals with such issues is Functional Analysis.)

Anyway, having accepted (2), we can do one more step and consider shift by 2, which is the composition of two unit shifts:

$$T^2 f(x) = f(x+2); \qquad T^2 = (e^D)^2 = e^{2D}.$$

(The new formula is consistent with Taylor's expansion, again.) In general, shift by any positive value s can be considered as operator

$$T^s f(x) = f(x+s), \qquad T^s = (e^D)^s = e^{sD}. \tag{3}$$

3 Summation of series in terms of operator D

According to Eq. (3), we can formally write

$$\begin{aligned} f(x) + f(x+1) + f(x+2) + \ldots \;&=\; \sum_{n=0}^{\infty} f(x+n) \;=\; (1 + T + T^2 + \ldots) f(x) \;=\; \frac{1}{1-T} f(x) \\ &=\; \frac{1}{1-e^D} f(x). \end{aligned} \tag{4}$$

Now, if D was an ordinary variable, we would have from Eq. (1)

$$\begin{aligned} \frac{1}{1-e^D} \;&=\; \frac{-1}{D}\,\frac{D}{e^D - 1} \;=\; \frac{-1}{D}\left(1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^k\right) \\ &=\; \frac{-1}{D} + \frac{1}{2} - \sum_{k=2}^{\infty} \frac{B_k}{k!} D^{k-1}. \end{aligned} \tag{5}$$

The terms in $\sum$ on the right are constant multiples of powers of the operator D, like in Taylor's formula. The "constant term" $\frac{1}{2}$ is simply the operator of multiplication by $1/2$.

The only question, from the formal point of view, is what does $1/D$ mean? Naturally, we anticipate that it is an operator satisfying the relation

$$D \cdot \frac{1}{D} = 1$$

where the "multiplication" $\cdot$ means composition of operators, and 1 in the RHS stands for the identity operator. Therefore, $1/D$ is to be an inverse operator to differentiation, that is integration. So $(1/D)f(x)$ will be an antiderivative of $f(x)$. Since there are many antiderivatives (differing by an additive constant), there remains some indeterminancy.

Euler makes a distinguished choice: he choses infinity as the formal lower limit of integration:

$$\frac{1}{D} f(x) = \int_{\infty}^{x} f(t)\, dt,$$

resp.

$$\frac{-1}{D} f(x) = \int_{x}^{\infty} f(t)\, dt. \tag{6}$$

Substituting to (5) and applying the "operator expansion" (5) to the function f, we obtain according to (4):

$$f(x) + f(x+1) + f(x+2) + \ldots = \int_{x}^{\infty} f(t)\, dt + \frac{f(x)}{2} - \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x). \tag{7}$$

This is a pure formal Euler-Maclaurin series. In many cases RHS diverges, but its truncation gives a good approximation to the difference between the sum and the integral. Explicitly, we have

$$\sum_{k=0}^{\infty} f(x+k) - \int_{x}^{\infty} f(t)\, dt \;=\; \frac{f(x)}{2} - \frac{f'(x)}{12} + \frac{f'''(x)}{30 \cdot 24} - \frac{f^{(5)}(x)}{42 \cdot 720} - \frac{f^{(7)}(x)}{30 \cdot 40320} + \ldots.$$

4 The Euler-Maclaurin formula with remainder term

As a reference, here is one possible *exact* form of the E-M formula, somewhat similar to Taylor's formula with remainder in integral form. We can identify the whole left side as the error term for the composite trapezoidal rule.

$$\int_{x}^{x+n} f(t)\, dt - \frac{f(x) + f(x+n)}{2} - \sum_{k=1}^{n-1} f(x+k) \;=\; \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} \left(f^{(2j-1)}(x) - f^{(2j-1)}(x+n) \right)$$
$$- \frac{-1}{(2m+1)!} \int_{x}^{x+n} \beta_m(t-x)\, f^{(2m+1)}(t)\, dt,$$

where $\beta_m(\cdot)$ is a universal (independent of f and n) bounded periodic function with period 1. Like in many other error estimations, explicit expression for β_m has little impact. It is in fact piecewise-polynomial (though not a spline). In the segment $[0, 1]$,

$$\beta_m(x) = B_{2m+1}(x),$$

where $B_{2m+1}(x)$ is the so called $(2m+1)$-th Bernoulli polynomial, appearing as an expression of sum of $2m$-th powers of integers from 1 to x, cf. formula (1) in AMAT-2130, Lab.2A (see weblink at the beginning).