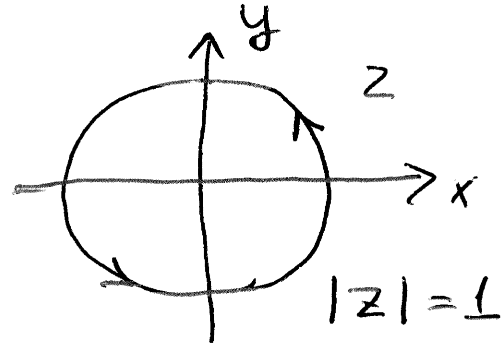
14.7.6

$$I = \int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

New integration variable

$$z = e^{i\theta}\,; \quad dz = i e^{i\theta} d\theta = i z\, d\theta \rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$



$$I = \frac{1}{2i}\oint_{|z|=1} \frac{dz}{z\left(2 + \frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2}$$

$$= \frac{4}{2i}\oint \frac{z\,dz}{(z^2+4z+1)^2}$$

The zeros of the denominator are the poles of the integrand:

$$z_{1,2} = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$$

$z_1 = -2-\sqrt{3}$ – this zero is outside of the integration contour $|z|=1$. Only $z_2 = -2+\sqrt{3}$, $|z_2| < 1$ is inside the contour

Thus

$$I = 2\pi i \cdot \frac{4}{2i} \cdot \mathrm{Res}\left(z = -2+\sqrt{3}, \frac{z}{(z^2+4z+1)}\right)$$

$z_2 = -2+\sqrt{3}$ — pole of the second order

$$\mathrm{Res}(z_2) = \left(\frac{d}{dz} \frac{z\cancel{(z-z_2)^2}}{(z-z_1)^2\cancel{(z-z_2)^2}} \right)_{z=z_2} =$$

$$= \frac{1}{(z_2-z_1)^2} - \frac{2z_2}{(z_2-z_1)^3} =$$

$$= \frac{1}{(2\sqrt{3})^2} - \frac{2(-2+\sqrt{3})}{(2\sqrt{3})^3} = \frac{4}{(2\sqrt{3})^3} = \frac{1}{6\sqrt{3}}$$

$$I = \cancel{2}\pi \cancel{i} \cdot \frac{4}{\cancel{2i}} \cdot \frac{1}{6\sqrt{3}} = \boxed{\frac{2\pi}{3\sqrt{3}}}$$

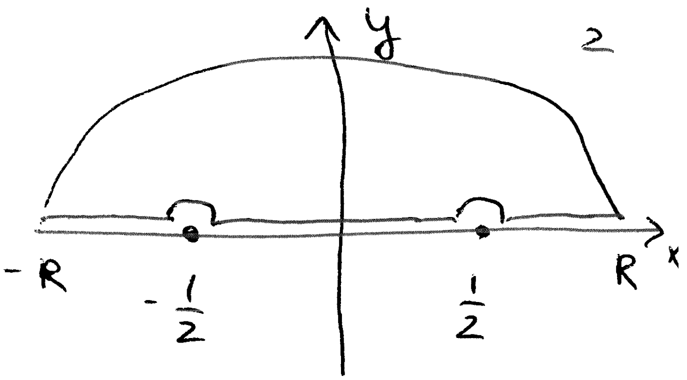
14.7.27

$$I = \int_0^\infty \frac{\cos \pi x}{1-4x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos \pi x}{1-4x^2}$$

$$= \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{1-4x^2}\, dx$$

Let us consider the following integral:

$$\oint_C \frac{e^{i\pi z}}{1-4z^2}\, dz$$



$R \to \infty$

The integrand is analytic in the upper half of the complex plane, thus

$$\oint \frac{e^{i\pi z}}{1-4z^2}\, dz = 0$$

On the other hand,

$$\oint \frac{e^{i\pi z}}{1-4z^2}\, dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{1-4x^2} - \frac{1}{2}\, 2\pi i \operatorname{Res}(-\tfrac{1}{2})$$
$$- \frac{1}{2} \cdot 2\pi i \operatorname{Res}(\tfrac{1}{2})$$

$$\mathrm{Res}\left(z=\frac{1}{2},\ \frac{e^{i\pi z}}{1-4z^2}\right) = \left.\frac{e^{i\pi z}}{\frac{d}{dz}(1-4z^2)}\right|_{z=\frac{1}{2}} =$$

$$= \left.\frac{e^{i\pi z}}{-8z}\right|_{z=\frac{1}{2}} = \frac{e^{\frac{1}{2}i\pi}}{-4} = -\frac{i}{4}$$

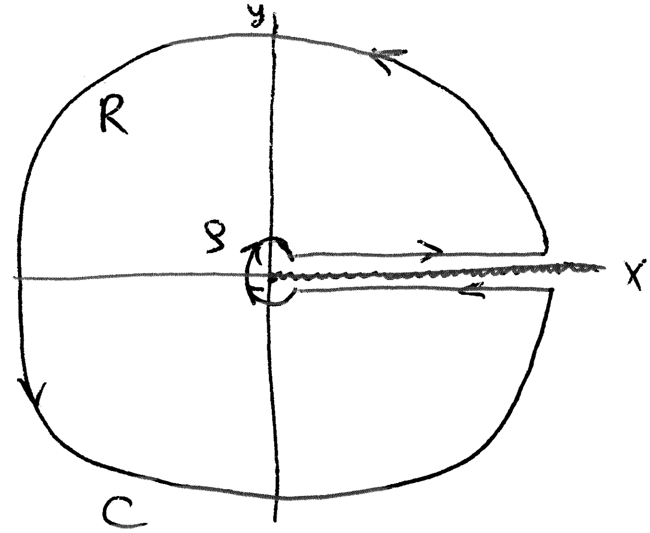
$$\mathrm{Res}\left(z=-\frac{1}{2},\ \frac{e^{i\pi z}}{1-4z^2}\right) = \left.\frac{e^{i\pi z}}{-8z}\right|_{z=-\frac{1}{2}}$$

$$= \frac{e^{-i\pi\frac{1}{2}}}{4} = -\frac{i}{4}$$

Finally,

$$I = \frac{1}{2}\,\mathrm{Re}\left[\pi i\left(-\frac{i}{4}-\frac{i}{4}\right)\right] = \boxed{\frac{\pi}{2}}$$

14.7.33



$$I = \int_0^{\pi} \frac{\sqrt{x}}{1+x^2}\,dx$$

Let us consider the integral:

$$\oint_C \frac{\sqrt{z}}{1+z^2}\,dz = I + \underbrace{\int_R}_{=0} + I' + \underbrace{\int_\rho}_{=0}$$

where I' is integral along the bottom side of the branch cut (along the real axis)

$$\sqrt{z} = \sqrt{x e^{2\pi i}} = \sqrt{x}\, e^{\pi i} = -\sqrt{x}$$

$$I' = \int_\infty^0 \frac{dx\,(-\sqrt{x})}{1+x^2} = \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = I$$

Thus,

$$\oint_C \frac{\sqrt{z}}{1+z^2}\,dz = 2I$$

On the other hand,

$$\oint_C \frac{\sqrt{z}}{1+z^2}\,dz = 2\pi i\left(\mathrm{Res}(i) + \mathrm{Res}(-i)\right)$$

$$\mathrm{Res}\left(z=i, \frac{\sqrt{z}}{1+z^2}\right) = \left.\frac{\sqrt{z}}{\frac{d}{dz}(1+z^2)}\right|_{z=i} =$$

$$= \left.\frac{\sqrt{z}}{2z}\right|_{z=i} = \left.\frac{1}{2}\frac{1}{\sqrt{z}}\right|_{z=e^{\frac{\pi i}{2}}} = \frac{1}{2}e^{-\frac{\pi i}{4}}$$

$$\mathrm{Res}\left(z=-i, \frac{\sqrt{z}}{1+z^2}\right) = \left.\frac{1}{2}\frac{1}{\sqrt{z}}\right|_{z=e^{\frac{3}{2}\pi i}} =$$

$$= \frac{1}{2}e^{-\frac{3}{4}\pi i} = -\frac{1}{2}e^{+\frac{\pi i}{4}}$$

Finally,

$$2I = 2\pi i\left(\frac{1}{2}e^{-\frac{\pi i}{4}} - \frac{1}{2}e^{\frac{\pi i}{4}}\right) =$$

$$= 2\pi i\cdot(-i)\cdot\sin\frac{\pi}{4} = 2\pi\frac{1}{\sqrt{2}}$$

$$\boxed{I = \frac{\pi}{\sqrt{2}}}$$

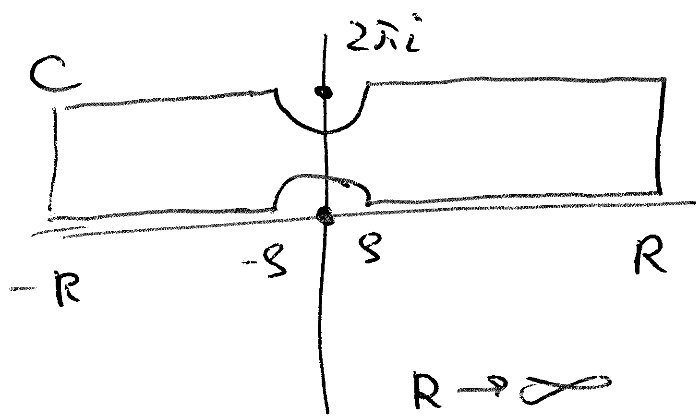
d to thor!

14.7.38 $\qquad 0<p<1$

$$I = \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx$$

Let us consider the following contour integral

$$\oint_C \frac{e^{pz}}{1-e^{z}}\,dz$$



$R\to\infty$
$\rho\to 0$

The function is analytic inside the contour, thus

$$\oint_C \frac{e^{pz}}{1-e^{z}}\,dz = 0$$

On the other hand,

$$\oint_C = \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx + \int_{\infty}^{-\infty} \frac{e^{p(x+2\pi i)}}{1-e^{x+2\pi i}}\,dx$$

$$-\frac{1}{2}\cdot 2\pi i\,\mathrm{Res}(0) - \frac{1}{2}\cdot 2\pi i\,\mathrm{Res}(2\pi i)$$

$$\int_{\infty}^{-\infty} \frac{e^{px}e^{2\pi p i}}{1-e^{x}}\,dx = -e^{2\pi p i}\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx = -e^{2\pi p i}\,I$$

$$\mathrm{Res}\left(z=0, \frac{e^{pz}}{1-e^{z}}\right) = \left.\frac{e^{pz}}{\frac{d}{dz}(1-e^{z})}\right|_{z=0} =$$

$$= -\left.\frac{e^{pz}}{e^{z}}\right|_{z=0} = -1$$

$$\mathrm{Res}\left(z=2\pi i, \frac{e^{pz}}{1-e^{z}}\right) = -\left.\frac{e^{pz}}{e^{z}}\right|_{z=e^{2\pi i}}$$

$$= -e^{2\pi p i}$$

$$I\left(1-e^{2\pi p i}\right) = \frac{1}{2}\cdot 2\pi i\left(-1-e^{2\pi p i}\right)$$

$$I = -\pi i\,\frac{1+e^{2\pi p i}}{1-e^{2\pi p i}} = -\pi i\,\frac{e^{-\pi p i}+e^{\pi p i}}{e^{-\pi p i}-e^{\pi p i}} =$$

$$= -\pi i\,\frac{2\cos \pi p}{2i\sin(-\pi p)} = \boxed{\pi\,\frac{\cos \pi p}{\sin \pi p}}$$