MEAN-FIELD THEORY OF FERROMAGNETISM

Fall semester 2025

https://www.phys.uconn.edu/~rozman/Courses/P2200_25F/

Last modified: November 11, 2025

Mean-field approximation

Consider a lattice containing a spin s at each site that can point either up ($s = +1$) or down ($s = -1$). Each spin interacts with its close neighbors on the lattice. The energy of the system is as follows:

$$E_N = -J \sum_{i=1}^{N} s_i \sum_{\langle ij \rangle} s_j. \tag{1}$$

Here $J > 0$, is the coupling constant, so the energy is minimized when the neighboring spins point in the same direction; $\langle ij \rangle$ denotes summation over neighbors of spin i; N is the number of sites on the lattice.

Such a model magnetic material has two distinct phases, namely a *paramagnetic* phase in which spins are disordered due to thermal fluctuations, and a *ferromagnetic* phase in which spins spontaneously are aligning in one direction.

We can quantitatively distinguish two phases by defining the *magnetization*

$$m = \frac{1}{N} \sum_{i=1}^{N} \langle s_i \rangle, \tag{2}$$

where $\langle \ldots \rangle$ denotes the thermal averaging. Since all spins are identical, $\langle s_i \rangle = \langle s \rangle$, the magnetization is equal to the average value of a spin:

$$m = \langle s \rangle. \tag{3}$$

The paramagnetic and ferromagnetic phases are separated by a phase transition at a critical temperature $T = T_c$. The system is in the paramagnetic state, $m = 0$, if the temperature $T > T_c$, and it is in the ferromagnetic state, $m \neq 0$, if $T < T_c$.

We assume that each spin interacts in the same way with its neighbors. We can replace the value of each spin by its average value plus fluctuations.

$$s_i \longrightarrow \langle s\rangle + \delta_i. \tag{4}$$

If we assume that

$$\sum_{\langle ij\rangle} \delta_j \ll \sum_{\langle ij\rangle} \langle s\rangle, \tag{5}$$

we can neglect the fluctuations. The energy of the system can be rewritten as follows:

$$E_N = -\sum_{i=1}^{N} s_i \left(J \sum_{\langle ij\rangle} \langle s\rangle \right) = -B_{\text{eff}} \sum_{i=1}^{N} s_i, \tag{6}$$

where

$$B_{\text{eff}} = J\, z \langle s\rangle, \tag{7}$$

and z is the number of nearest neighbors.

The expression Eq. (6) is the energy of *non-interacting spins* in the effective magnetic field B_{eff}.

For non-interacting spins in the field B_{eff}, the thermal averaged value of the spin, $\langle s\rangle$, is as follows:

$$\langle s\rangle = \tanh\left(\frac{B_{\text{eff}}}{k_B T}\right) = \tanh\left(\frac{Jz}{k_B T}\langle s\rangle\right), \tag{8}$$

where T is the temperature of the system, k_B is the Boltzmann constant.

Introducing the notation T_c for the so called *critical temperature*,

$$T_c \equiv \frac{Jz}{k_B}, \tag{9}$$

we can write Eq. (8) in the following universal form:

$$m = \tanh\left(\frac{T_c}{T} m\right). \tag{10}$$

Eq. (10) *is the mean-field* equation for the magnetisation.

Let's investigate the solution of Eq. (10). We rewrite the equation in the following form:

$$\tau u = \tanh(u), \tag{11}$$

where where we introduced the notation

$$\tau = \frac{T}{T_c} \tag{12}$$

and a new variable u,

$$u = \frac{m}{\tau}, \quad m = \tau u. \tag{13}$$

In Fig. 1 we plotted the graphs of the left and right hands sides of Eq. (11) as functions of u for several values of τ.

If $\tau > 1$, i.e. $T > T_c$, the graphs do not cross (except for trivial solution $u = 0$). That means that for temperatures $T > T_c$ the magnetization of the system is zero, and the system is in paramagnetic phase.

For temperatures below T_c, the graphs do intersect at $u(T) \neq 0$. That means that the system is spontaneously magnetized when $T < T_c$.

The critical temperature T_c below which the system becomes spontaneously magnetized without any external magnetic fields is therefore given by Eq. (9). The results of numerical solution of Eq. (10) for the magnetization m as a function of temperature are presented in Fig. 2.

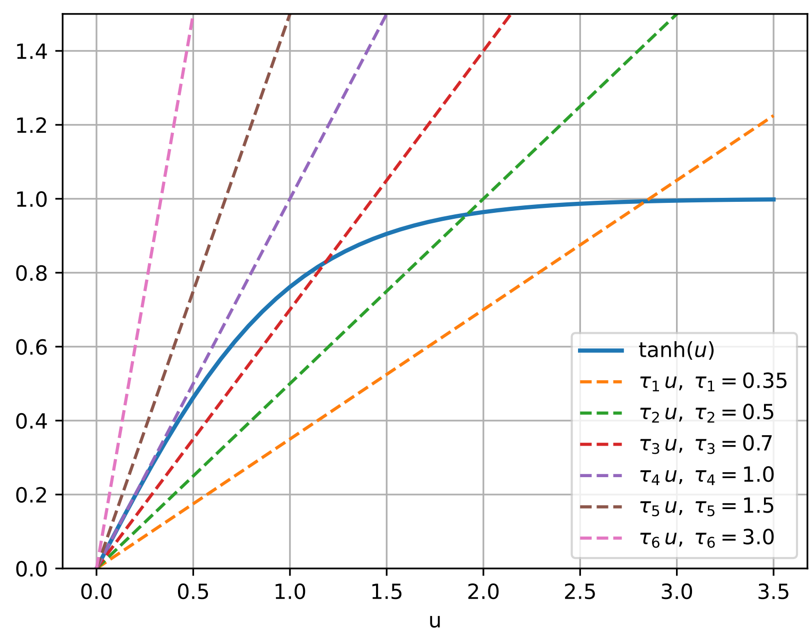

Figure 1: Graphical solution of Eq. (10).

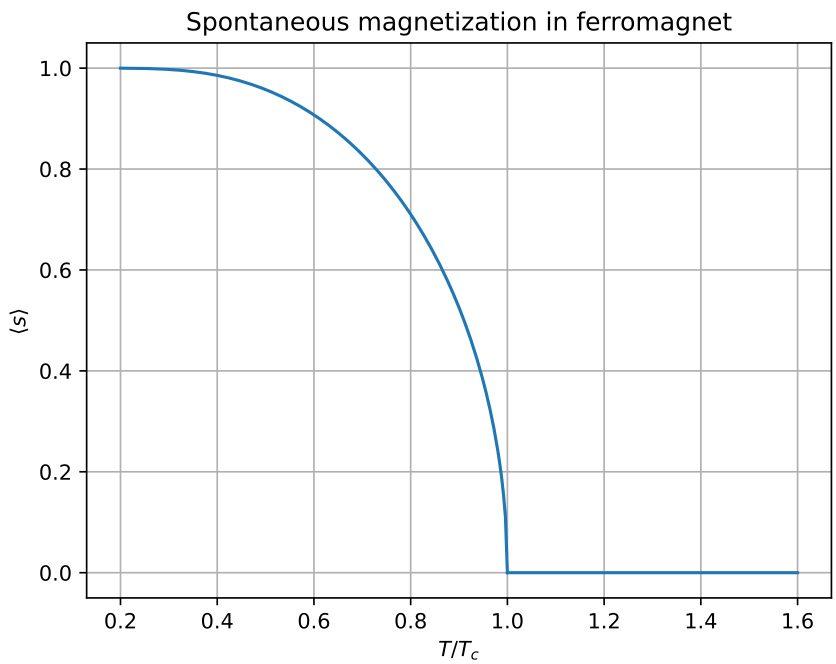

Figure 2: Numerical solution of Eq. (10).