MONTE CARLO CALCULATIONS OF THE BASE OF NATURAL LOGARITHM

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https://www.phys.uconn.edu/~rozman/Courses/P2200_25F/

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Consider the following Monte Carlo algorithm for calculating the value of e, the base of natural logarithm:

> Let X_i be independent random numbers from the uniform distribution on $[0,1)$. Let n be the minimum number such that $\sum_{i=1}^{n} X_i > 1$. (Note that n is also a random variable; $n \geq 2$.) Then, the average value of n is equal to e.

The proof of the statement is as follows:

Let $\Pr(n)$ be the probability distribution for the random variable n:

$$\Pr(n) \equiv P(X_1 + \ldots + X_{n-1} \leq 1 \text{ and } X_1 + \ldots + X_n > 1). \tag{1}$$

First, let's determine the probability $P(X_1 + \ldots + X_n \leq 1)$. We prove by induction the following more general result: if $0 \leq t \leq 1$, then

$$P(X_1 + \ldots + X_n \leq t) = \frac{t^n}{n!}. \tag{2}$$

The base case, $n = 1$, is correct: for the uniform distribution on $[0,1)$ $P(X \leq t) = t$.

If Eq. (2) holds for n, then $P(X_1 + \ldots + X_{n+1} \leq t)$ is as follows:

$$P(S_{n+1} \leq t) = \int_0^t P(S_n + x \leq t)\,\rho(x)\,\mathrm{d}x = \int_0^t P(S_n \leq t - x)\,\rho(x)\,\mathrm{d}x = \int_0^t \frac{(t-x)^n}{n!}\,\mathrm{d}x = \frac{t^{n+1}}{(n+1)!}, \tag{3}$$

where S_n denotes the sum $X_1 + \cdots + X_n$, and for uniform distribution $\rho(x) = 1$.

Hence,

$$P(S_n \leq 1) = \frac{1}{n!}. \tag{4}$$

Next,

$$P(S_n > 1 \text{ and } S_{n-1} < 1) = P(S_{n-1} < 1) - P(S_n < 1). \tag{5}$$

Therefore,

$$\Pr(n) = \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!}. \tag{6}$$

Finally, the average value of n,

$$\bar{n} \equiv \sum_{n=2}^{\infty} n \Pr(n) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e. \tag{7}$$

References

[1] K. G. Russell. “Estimating the Value of e by Simulation”. In: *The American Statistician* 45.1 (1991), pp. 66–68.