HW08

due November 19, 2025

Accept the assignment in GitHub Classroom and launch the codespace. To work on the problems, open the provided templates of Jupyter notebooks. Install Julia packages as required. Do not modify the *ready-to-run* code provided in the notebooks.

Show all your work and indicate your reasoning in order to receive the credit.

Name: ______________________________

Date: ______________________________

Collaborators: ____

(If applicable, the collaborators submit their individually written assignments together)

GitHub username: ___________________________

Question:	1	2	Total
Points:	70	5	75
Score:			

Instructor/grader comments:

1. (70 points) **Buckling instability and large deformations of slender rods**

   Buckling instability is a sudden change of the shape of a straight rod that is compressed longitudinally. Buckling does not happen until the compressive forces on the road terminals exceed a certain threshold, called *buckling threshold*.

   The buckling threshold of a slender rod, first determined by Euler, is as follows.

   $$F_B = \frac{\pi^2 EI}{L^2}.$$

   Here L is the length of the rod, E is the Young's modulus of the rod material, and I is the area moment of inertia of the cross section of the rod.

   For a wooden stick of length $L = 1$ m and circular cross section of diameter $D = 2$ cm, with wood Young's modulus $E = 10^{10}$ Pa, the buckling threshold is $F_B = 775$ N, corresponding to the weight of mass $m = 79$ kg.

   A stringed bow may be viewed as a straight rod that has been brought beyond the buckling threshold and is kept in mechanical equilibrium by the tension in the bow string. In this case, the deflection of the rod from its non-deformed equilibrium is not small compared to the dimension of the bow, but the strains in the material are still small. This permits using the linear elasticity theory to analyze large deflections of the bow.

   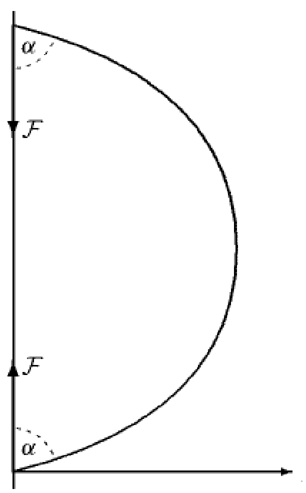

   Figure 1: The geometry of a stringed bow with opening angle α. The bow is kept in mechanical equilibrium by a force $\mathcal{F}$.

   The results of that theoretical analysis of elastic deformation of the bow are as follows:

   i. The deformation of the bow can be described by a single parameter — the so called *opening angle*, α. See Fig. 1 for the sketch of a bow geometry and the definition of opening angle.

ii. The tension in the bow string, F, measured in units of of the buckling threshold, F_B, $f(\alpha) = \frac{F}{F_B}$, is as follows:

$$f(\alpha) = \frac{1}{\pi^2} I_1^2(\alpha), \tag{1}$$

where

$$I_1(\alpha) = \sqrt{2} \int_0^\alpha \frac{\mathrm{d}x}{\sqrt{\cos(x) - \cos(\alpha)}} \tag{2}$$

iii. The separation between the bow terminals, measured in units of the length of the bow, $z = \frac{Z}{L}$, is as follows:

$$z(\alpha) = \frac{I_2(\alpha)}{I_1(\alpha)}, \tag{3}$$

where

$$I_2(\alpha) = \sqrt{2} \int_0^\alpha \frac{\cos(x)\,\mathrm{d}x}{\sqrt{\cos(x) - \cos(\alpha)}}. \tag{4}$$

In Eqs. (1)–(4), α is the angle in radians.

The goals of the assignment are:

I. Evaluate the integrals Eqs. (2), (4) numerically. Use the results to plot the following graphs: (a) the tension in the bow string vs. the bow's opening angle, (b) the distance between the bow terminals vs. the bow's opening angle, and (c) the tension in the bow string vs. the distance between the bow terminals.

II. Solve nonlinear equation $z(\alpha) = 0$ and determine the opening angle and the tension in the bow string that are required to bring the bow terminals together.

(a) The integral in Eqs. (2) and (4) are written in a form that is not suitable for accurate numerical evaluations: the term $\cos(x) - \cos(\alpha)$ causes catastrophic cancellations in the denominators of the integrands when $x \to \alpha$.

One way to avoid the catastrophic cancellations, is to rewrite the integrand in a form that doesn't contain a subtraction of very close floating point values.

Rewrite integrals I_1, I_2 in a form free from catastrophic cancellations.

Hints: first use the trigonometric identity

$$\cos(x) - \cos(\alpha) = 2\sin\left(\frac{\alpha + x}{2}\right)\sin\left(\frac{\alpha - x}{2}\right). \tag{5}$$

Next, introduce a new integration variable, y,

$$y = \alpha - x, \quad 0 \leq y \leq \alpha, \quad x = \alpha - y, \quad \mathrm{d}x = -\mathrm{d}y, \quad \alpha + x = 2\alpha - y. \tag{6}$$

Write your derivations in the space below.

(b) Write Julia functions, `I1(alpha)` and `I2(alpha)`, that accept the value of the opening angle of the bow (in radians), and return the numerical value of the integrals $I_1(\alpha)$ and $I_2(\alpha)$ respectively. Use Julia package `QuadGK` for numerical evaluation of the integrals.

(c) In three separate figures, plot the graphs $f(\alpha)$, $z(\alpha)$ and $f(z)$ for $5° \leq \alpha \leq 150°$ (angle in degrees). Use at least 30 data points. For conversion from degrees to radians use the following relation:

$$\alpha_{\text{rad}} = \frac{\pi}{180}\alpha_{\text{deg}} \tag{7}$$

Provide axes labels, grids, titles for all your graphs.

See Fig. 2 for a possible graph of $f(\alpha)$ that your code is expected to produce.

Use your graph to determine (approximately) the dimensionless tension in the bow string for the opening angle 100°. What is the value of string tension, in Newtons, for a bow made from the wooden stick that was discussed earlier?

(d) Solve the equation $z(\alpha) = 0$ numerically to find the opening angle corresponding to touching terminals of the bow. Use Julia package `Roots`. Use bisection method. Determine the initial interval for bisection by the visual inspection of the plot $z(alpha)$. Determine the value of tension in the string, in Newtons, that is required to bring the rod terminals together.

2. (5 points) **Submission of the assignment.**

Comment out the commands installing Julia packages (but do not delete them). Clear all output cells in your Jupyter notebook(s), save and close them. Delete any unnecessary or temporary notebooks you created (e.g., `Untitled.ipynb`). Commit all your code changes, making sure to include updates to Project.toml and Manifest.toml. Finally, push your commits to the assignment's GitHub repository.

The submitted code must run without producing any warnings or error messages (unless requested by the problem statement).

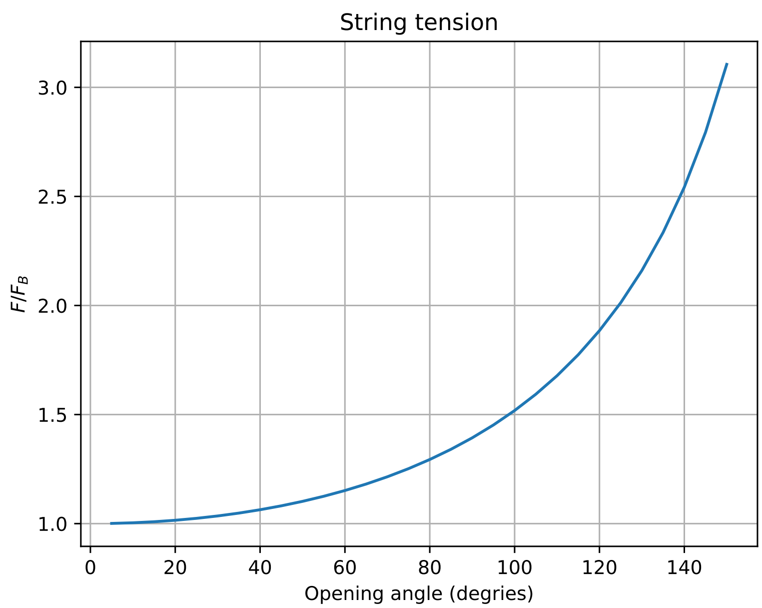

Figure 2: Expected graph for dimensionless bow string tension, $f(\alpha)$.