HW07

due November 12, 2025

Accept the exam assignment in GitHub Classroom and launch the codespace. To work on the problems, open the provided templates of Jupyter notebooks. Install Julia packages as required. Do not modify the *ready-to-run* code provided in the notebooks.

Show all your work and indicate your reasoning in order to receive the credit.

Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together, in class, in person)

GitHub username: ___________________________

Question:	1	2	3	4	Total
Points:	25	20	25	5	75
Score:					

Instructor/grader comments:

1. (25 points) **The internal structure of white dwarfs**

   White dwarfs are the final evolutionary state of stars whose mass is not high enough to become a neutron star or a black hole.

   The following system of 2 dimensionless differential equations describing the radial distribution of the density, $\rho(r)$ and mass, $m(r)$, inside a white dwarf star:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \rho r^2, \tag{1}$$
$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{m\rho}{\gamma(\rho) r^2}. \tag{2}$$

   Here

$$\gamma(\rho) = \frac{\rho^{2/3}}{3\sqrt{1+\rho^{2/3}}}. \tag{3}$$

   In the equations above the density is measured in units of ρ_0,

$$\rho_0 = \frac{M_p\, m_e^3\, c^3}{3\,\pi^2\,\hbar^3\, Y_e} = 9.82\times 10^8\, Y_e^{-1}\ \mathrm{kg\,m^{-3}}., \tag{4}$$

   the distances are measured in units of R_0,

$$R_0 = \left(\frac{m_e\, c^2\, Y_e}{4\,\pi\,\rho_0\, G\, M_p}\right)^{\frac{1}{2}} = 7.71\times 10^6\, Y_e\ \mathrm{m}, \tag{5}$$

   and the mass is measured in units of M_0

$$M_0 = 4\,\pi\, R_0^3\,\rho_0 = 5.66\times 10^{30}\, Y_e^2\ \mathrm{kg}. \tag{6}$$

   Here M_p is the mass of the proton, m_e is the mass of of the electron, Y_e is the number of electrons per nucleon, c is the speed of light, $\hbar$ is the Planck constant.

   Once the internal density of the white dwarf, $\rho(r)$, is calculated, the internal pressure can be found using the relation:

$$P(\rho) = \frac{1}{8}\left(x\left(2x^2-3\right)\sqrt{x^2+1} + 3\sinh^{-1}(x)\right), \qquad x = \rho^{1/3}. \tag{7}$$

   In Eq. (7) the pressure is measured in units of

$$P_0 = \rho_0\, Y_e\, \frac{m_e}{M_p}\, c^2 = 4.8\times 10^{22}\,\mathrm{Pa}. \tag{8}$$

   For reference, the standard atmospheric pressure at sea level is $P_{\mathrm{atm}} \approx 10^5$ Pa.

(a) Solve the differential equations describing the internal structure of white dwarfs for $\rho_c = 30000.0$.

(b) Plot plot your results for $\rho(r)$ and $P(r)$. The expected graphs are shown in figures 1 and 2.

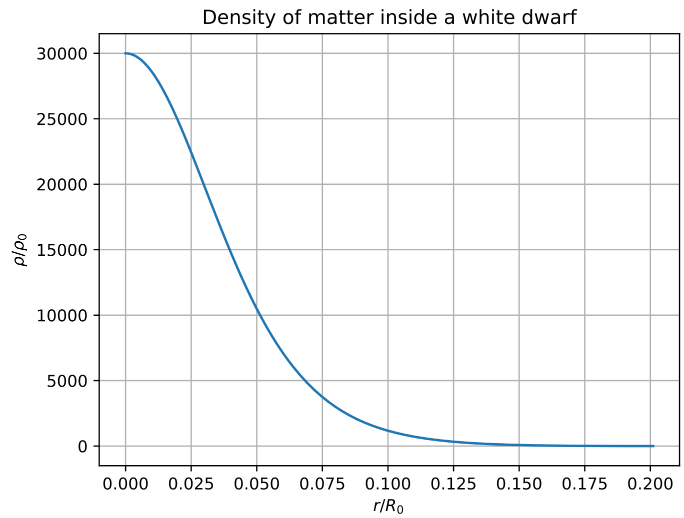

Figure 1: Expected graph of density of matter inside a white dwarf.

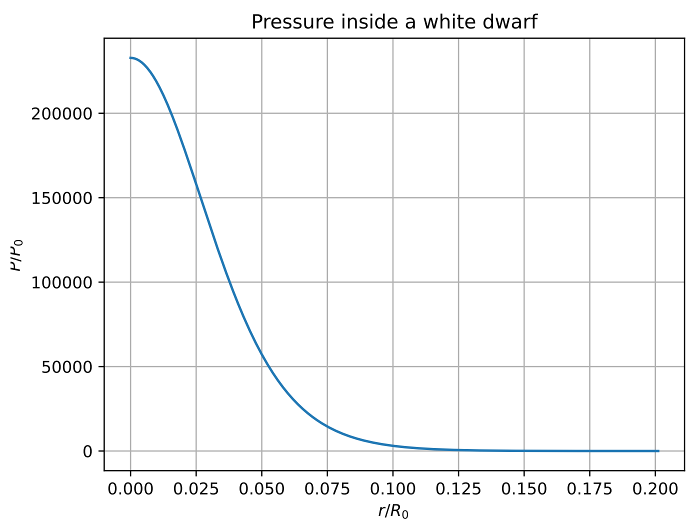

Figure 2: Expected graph of pressure inside a white dwarf.

2. (20 points) **An example of catastrophic cancellation**

Recall that when we investigated errors of finite difference approximation to derivatives, we found that the errors are abnormally large in the limit when the finite difference spacing, $h \to 0$. The phenomenon of error growth is not restricted to calculation of derivatives. In fact, it is always present when calculations involve subtracting two nearby numbers. The phenomenon is known as *catastrophic cancellation*

The goal of the assignment is to conduct numerical experiments involving a mathematical expression with catastrophic cancellations, as well as observe that the expression can be rewritten in a form that mitigates the cancellation problem.

Consider the function $\phi(x)$:

$$\phi(x) = \frac{a^{3/2}}{\sin(x)} \left(\frac{1}{\sqrt{a-x}} - \frac{1}{\sqrt{a+x}} \right),$$

where $a = 10^7$. For reference, for small values of x, $x \ll 1$,

$$\phi(x) \approx 1 + \frac{x^2}{6}$$

(a) Plot the graph of $\phi(x)$ for equidistant values of x, $10^{-10} \leq x \leq 10^{-8}$. Plot the graph using at least 50 data points. Provide the title, grid, axes labels for your graph. Comment whether your graph behavior is close the the expected one $\phi(x) \approx 1$.

The expected plot is shown in Fig. 3.

(b) It is often possible to rewrite the expression that suffers from catastrophic cancellations in a form that is cancellation-free.

Consider the function:

$$\psi(x) = \frac{2a^{3/2} x}{\sin(x)\sqrt{a^2 - x^2}\left(\sqrt{a+x} + \sqrt{a-x}\right)}$$

On the same figure, for $0.1 \leq x \leq 2.5$, plot the graphs of $\phi(x)$ and $\psi(x)$. By visually inspecting the plots, confirm that $\phi(x)$ and $\psi(x)$ is the same function, just written in two different forms.

Use at least 50 data points for each of ϕ and ψ graphs. Provide the title, grid, axes labels, legend for your figure.

(c) Finally, plot the graphs of $\phi(x)$ and $\psi(x)$ for small values of x: $10^{-10} \leq x \leq 10^{-8}$. Use at least 50 data points. Provide the title, grid, axes labels, legend for your figure. The expected plot is shown in Fig. 4. Comment whether the function $\psi(x)$ suffers from catastrophic cancellations.

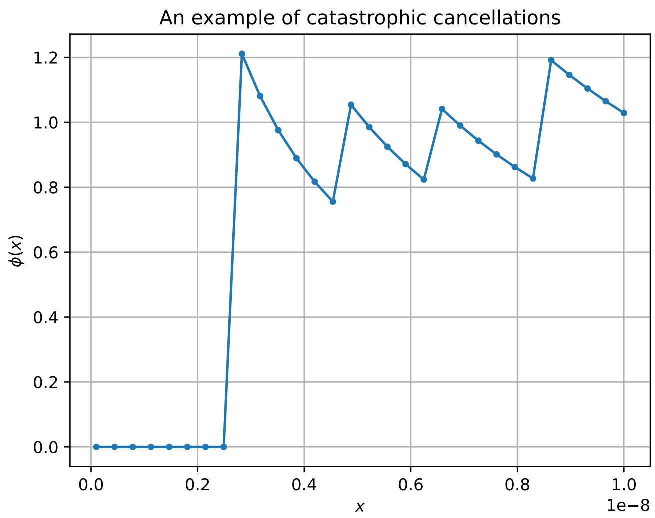

Figure 3: Expected plot for Problem 2(a).

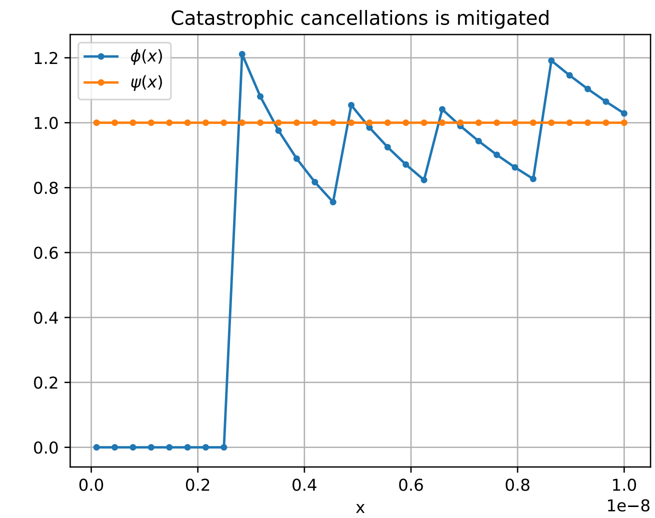

Figure 4: Expected plot for Problem 2(c).

3. (25 points) **Floating point numbers**

   Floating point numbers typically are represented in computers in the following binary form:

$$x = \pm\left(1 + \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_d}{2^d}\right) \times 2^E,$$

   where $b_i \in \{0,1\}$, $i = 1,\ldots,d$; E is an integer that can be positive, negative, or 0.

   (a) What is the (approximate) value of machine epsilon for a microprocessor that uses $d = 10$? Explain in the space below.

   (b) For the same microprocessor, how many floating point numbers x are there, such that $4 \leq x < 16$? Explain in the space below.

   (c) For the same microprocessor, assuming that the smallest possible value of E is -15 and the largest possible value of E is 16, what are (approximately) the smallest and the largest positive floating point number? Explain in the space below. Present your answers as a power of 2 and as a decimal number.

4. (5 points) **Submission of the assignment.**

   Clear all output cells in your Jupyter notebook(s), then save and close it. Delete any unnecessary or temporary notebooks you created (e.g., `Untitled.ipynb`). Commit all your code changes, making sure to include updates to Project.toml and Manifest.toml. Finally, push your commits to the assignment's GitHub repository.

   The submitted code must run without producing any warnings or error messages.