Name: ____________________

Date: ____________________

Collaborators: ______________________________________

(Collaborators submit their individually written assignments together, in class, in person)

GitHub username: ____________________

Question:	1	2	Total
Points:	70	5	75
Score:			

Instructor/grader comments:

Limit cycles in non-linear dissipative systems. Van der Pol equation.

1. (70 points) The second order non-linear autonomous differential equation

$$\frac{d^2x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + x = 0, \qquad \mu > 0, \tag{1}$$

is called *van der Pol equation*. The equation models a non-conservative system in which energy is added to and subtracted from. The sign of the "coefficient" in the damping term in Eq. (1), $\left(x^2 - 1\right)$ changes, depending whether $|x|$ is larger or smaller than one, describing the inflow and outflow of the energy.

The equation was originally proposed in the 1920s to describe stable oscillations in electrical circuits employing vacuum tubes. By now, the van der Pol equation (under different names) has a long history of being used in physical and biological sciences.

Van der Pol oscillator is an example of a system that exibits the so called *limit cycle*. A limit cycle is an isolated closed trajectory $\dot{x} = \dot{x}(x)$ in the phase space $(x, \dot{x})$. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting. Otherwise the limit cycle is unstable.

Stable limit cycles model systems, e.g. the beating of a heart, that exhibit self-sustained oscillations. These systems oscillate even in the absence of external periodic forcing. There is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena. They can't occur in linear systerns. Of course, a linear system, such as a linear differential equation, can have closed orbits – periodic solutions, but they won't be isolated. If $x(t)$ is a periodic solution, then so is $\alpha x(t)$ for any constant $\alpha \neq 0$. Hence $x(t)$ is surrounded by a 'family' of closed orbits. Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions. Any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

Limit cycles are only possible in systems with dissipation. System that conserve energy do not have isolated closed trajectories.

The goal of this homework assignment is to verify numerically the existence of a limit cycle of the van der Pol equation and investigate its stability.

   (a) Accept the assignment in GitHub Classroom, launch the codespace, open the template of the notebook for the assignment.

(b) Use Markdown to fill in the blanks in the equations in the introductory part of the notebook.

(c) Rewrite van der Pol equation as a system of first order differential equations. Show your work in the space below.

(d) Modify your code of the Runge-Kutta midpoint method from HW04 so that it can solve systems of first order ordinary differential equations.

(e) Write the code for the function `vanderpol(t, y)` that implements the right-hand side of the system of first order differential equations that you derived in Part (c).

(f) The numerical parameters and the initial conditions for the first calculations are given to you. Solve the equation, and plot the coordinate and the velocity of the oscillator as functions of time.

If another figure, plot a phase trajectory of the van der Pol oscillator. An expected result of this plot is shown in Fig. 1.

(g) As you can see in your first phase plot, as t is increasing, the phase trajectory is approching a limiting trajectory.

To investigate the stability of this limiting trajectory, Select the second initial conditions for a phase space trajectory that starts outside the limiting trajectory. Leave other numerical parameters unchanged.

(h) Run your second calculations and plot, in the same figure the phase trajectories for your first and second calculations.

(i) Are the limit cycles of the van der Pol equation stable or unstable? Present your results and describe your reasoning in your jupyter notebook.

All plots of this assignments require axes labels, grids, legends, ad titles. For all phase trajectories mart the initial positions.

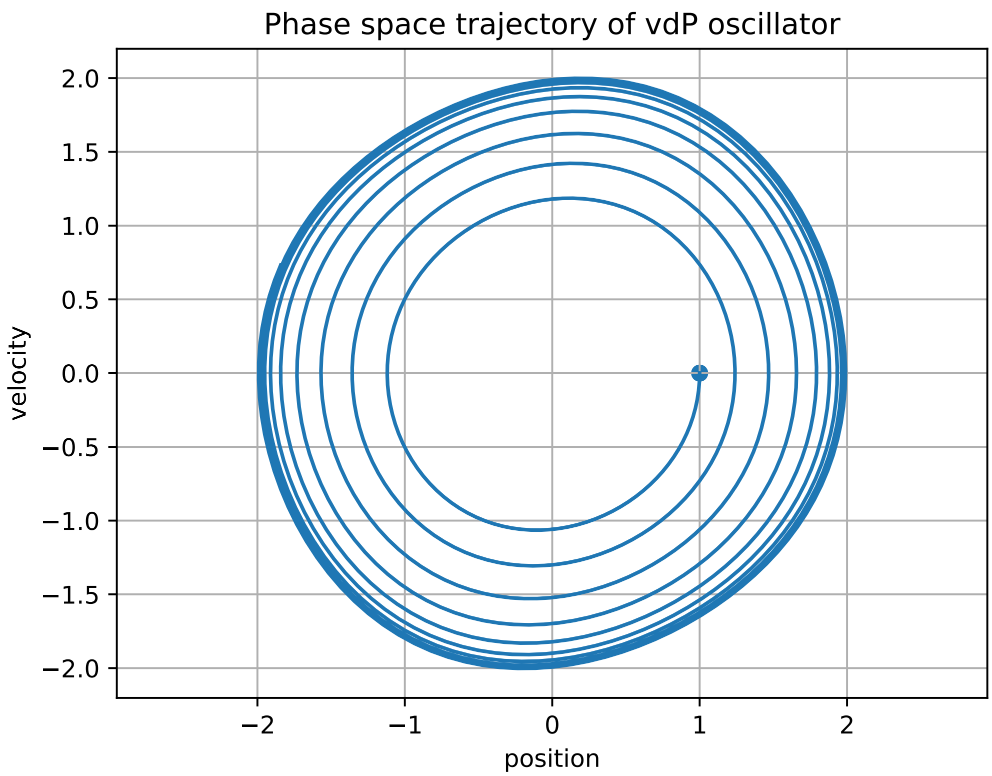

Figure 1: Typical phase trajectory of weakly-nonlinear van der Pol oscillator (colors online). Circular markers indicate the initial conditions.

2. (5 points) Clear all output cells in your Jupyter notebook(s), then save and close it. Delete any unnecessary or temporary notebooks you created (e.g., `Untitled.ipynb`). Commit all your code changes, making sure to include updates to Project.toml and Manifest.toml. Finally, push your commits to the assignment's GitHub repository. The submitted code must execute without producing any warnings or error messages.