Name: ______________________________

Date: ______________________________

Collaborators: ____

(If applicable, the collaborators submit their individually written assignments together)

Question:	1	2	3	4	5	Total
Points:	5	25	20	25	5	80
Score:						

Instructor/grader comments:

1. (5 points) Accept the assignment in GitHub Classroom, launch the codespace. To work with the assignments, open the templates of the provided notebooks.

2. **An example of catastrophic cancellation**

   Recall that when we investigated errors of finite difference approximation to derivatives, we found that the errors are abnormally large in the limit when the finite difference spacing, $h \to 0$. The phenomenon of error growth is not restricted to calculation of derivatives. In fact, it is always present when calculations involve subtracting two nearby numbers. The phenomenon is known as *catastrophic cancellation*

   We'll analyze the catastrophic cancellation in class later in the semester. For the time being, the goal of the assignment is to conduct numerical experiments involving a mathematical expression with catastrophic cancellations, as well as observe that the expression can be rewritten in a form that mitigates the cancellation problem.

   Consider the following expression:

   $$\phi(x) = \frac{a^{3/2}}{\sin(x)} \left(\frac{1}{\sqrt{a-x}} - \frac{1}{\sqrt{a+x}} \right),$$

   where $a = 10^7$. For reference, for $x \ll 1$,

   $$\phi(x) \approx 1.$$

   (a) (5 points) Plot the graph of $\phi(x)$ for equidistant values of x, $10^{-10} \leq x \leq 10^{-8}$. Plot the graph using at least 30 data points. Provide the title, grid, axes labels for your graph. Comment whether your graph behavior is close the the expected one $\phi(x) \approx 1$.

   The expected plot is shown in Fig. 1.

   (b) (10 points) It is often possible to rewrite the expression that suffers from catastrophic cancellations in a form that is cancellation-free. (We'll discuss how to do this later in the semester.)

   Consider the function:

   $$\psi(x) = \frac{2a^{3/2}x}{\sin(x)\sqrt{a^2 - x^2}\left(\sqrt{a+x} + \sqrt{a-x}\right)}$$

   On the same figure, for $0.1 \leq x \leq 2.5$, plot the graphs of $\phi(x)$ and $\psi(x)$. By visually inspecting the plots, confirm that $\phi(x)$ and $\psi(x)$ is the same function, just written in two different forms.

Use at least 50 data points for each of ϕ and ψ graphs. Provide the title, grid, axes labels, legend for your figure.

(c) (10 points) Finally, plot the graphs of $\phi(x)$ and $\psi(x)$ for small values of x: $10^{-10} \leq x \leq 10^{-8}$. Use at least 30 data points. Provide the title, grid, axes labels, legend for your figure. Comment whether the function $\psi(x)$ suffers from catastrophic cancellations.

The expected plot is shown in Fig. 2.

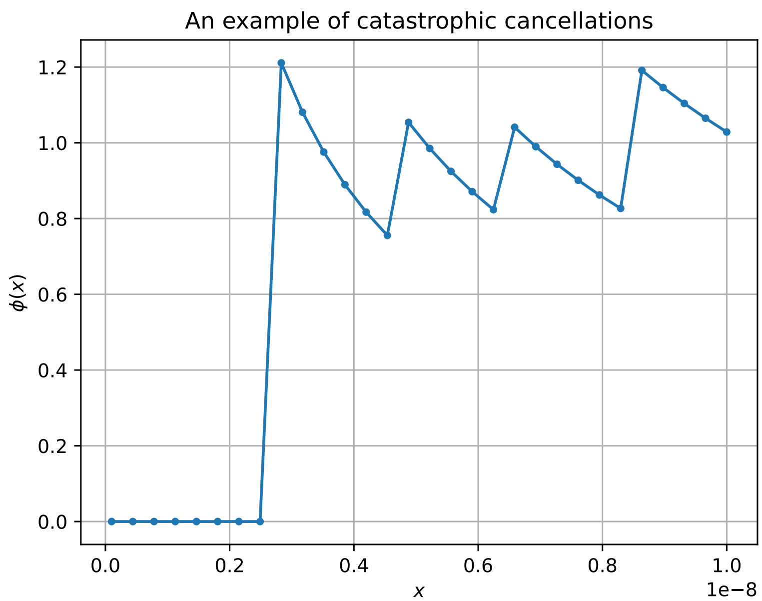

Figure 1: Expected plot for Problem 2(a).

3. **Automatic differentiation using dual numbers**

Julia programming language has a predefined function, `cbrt`, for the cube root of its argument. One of the advantages of using `cbrt` vs. using $x^{1/3}$, is that `cbrt` is defined for the negative arguments (whereas $x^{1/3}$ is not).

The goal of the assignment is to use dual numbers, as we introduce them in class, to "teach" Julia to calculate derivative of `cbrt`.

(a) (5 points) Plot the graph of `cbrt` for parameters given in the Jupyter notebook. Provide the axes labels, grid, title for your figure.

(b) (10 points) Use the implementation of Dual numbers as we develop in class.

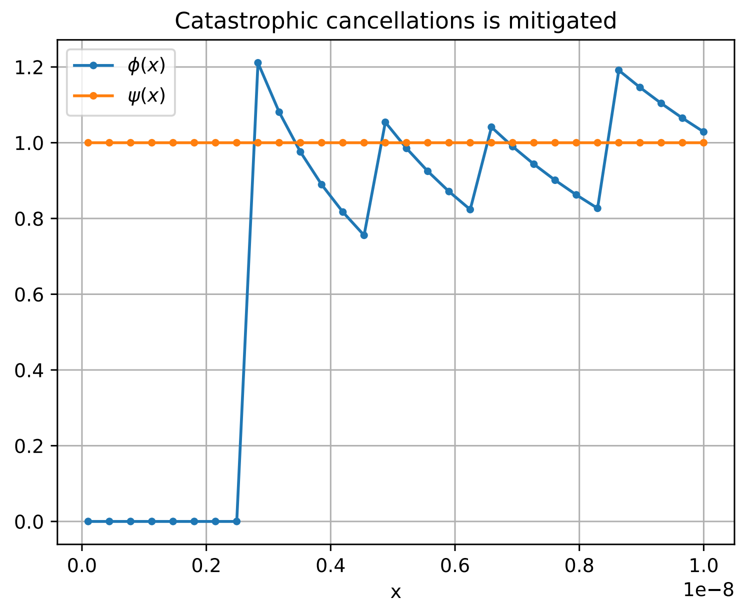

Figure 2: Expected plot for Problem 2(c).

Import the function `cbrt` and provide the rule the function to operate on Dual numbers. (You need only one rule.)

(c) (5 points) Follow the hints provided in the notebook and plot the graph of

$$w(x) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{x}}.$$

Provide the axes labels, grid, title.

The expected plot is shown in Fig. 3.

4. **Automatic differentiation and Newton's method**

In numerical analysis, the Newton's method, named after Isaac Newton, is an algorithm for funding a root of the equation

$$f(x) = 0.$$

The algorithm produces successively better approximations to the roots (or zeroes) of a real-valued function. The most basic version starts with a real-valued function $f(x)$,

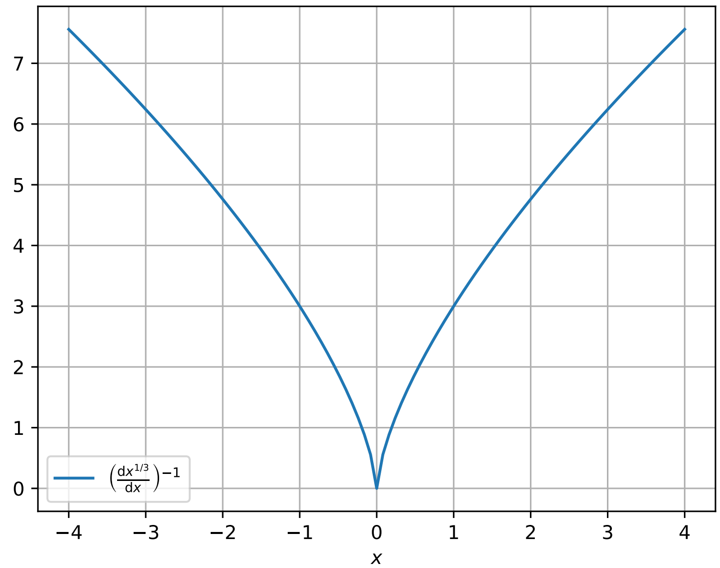

Figure 3: Expected plot for Problem 3(c) (color online).

its derivative $f'(x)$, and an initial guess x_0 for a root of $f(x)$. Better approximations are obtained using the foollowing iteration algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Julia provides several high-quality packages for calculation derivatives using automatic differentiation. For this assignment, you are required to use the package `ForwardDiff`. Once you loaded the package, you can calculate the value of the derivative of a function $g(s)$ at $s = x0$ using the following code fragment:

```
gderiv = ForwardDiff.derivative(g, x0)
```

The goal of the assignment is to modify the code for the Newton's method that we developed in class such that your code uses `ForwardDiff.derivative`, and solve the nonlinear equation $h(x) = 0$ where

$$h(x) = \tanh(\sinh(x) + cos(x)).$$

(a) (10 points) Modify the code for the Newton's method that we developed in class such that your code uses `ForwardDiff.derivative` functions instead of using Dual numbers.

(b) (5 points) Plot the graph of $h(x)$ to see the position of the root and estimate an initial approximation. Provide axes labels, grid, title. Describe your choice of the initial approximation.

(c) (10 points) Solve the nonlinear equation $h(x) = 0$ and plot the graph of $h(x)$ and the position of the root. Provide axes labels, grid, title, legend.

The expected plot is shown in Fig. 4.

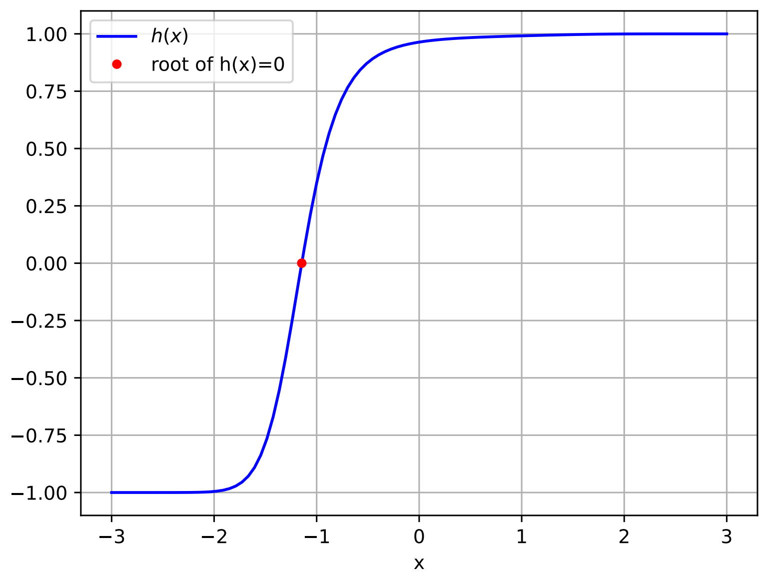

Figure 4: Expected plot for Problem 4(c) (color online).

5. (5 points) **Submission of the assignment.**

Clean the cells of your jupyter notebook(s), save and close the notebook(s). Delete unneeded notebooks if you created any (e.g. `Untitled.ipynb`). Commit all your changes to the project and push them to the assignment's GitHub repository.