Name: ______________________________

Date: ______________________________

Collaborators: ____

(If applicable, the collaborators submit their individually written assignments together)

Question:	1	Total
Points:	70	70
Score:		

Instructor/grader comments:

Limit cycles in non-linear dissipative systems. Van der Pol equation.

1. The second order non-linear autonomous differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad \varepsilon > 0 \tag{1}$$

is called *van der Pol equation*. The equation models a non-conservative system in which energy is added to and subtracted from. The sign of the "coefficient" in the damping term in Eq. (1), $\left(x^2 - 1\right)$ changes, depending whether $|x|$ is larger or smaller than one, describing the inflow and outflow of the energy.

The equation was originally proposed in the 1920s to describe stable oscillations in electrical circuits employing vacuum tubes. By now, the van der Pol equation (under different names) has a long history of being used in physical and biological sciences.

Van der Pol oscillator is an example of a system that exibits the so called *limit cycle*. A limit cycle is an isolated closed trajectory $\dot{x} = \dot{x}(x)$ in the phase space $(x, \dot{x})$. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting. Otherwise the limit cycle is in general unstable.

Stable limit cycles model systems, e.g. the beating of a heart, that exhibit self-sustained oscillations. These systems oscillate even in the absence of external periodic forcing. There is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena. They can't occur in linear systerns. Of course, a linear system, such as a linear differential equation, can have closed orbits – periodic solutions, but they won't be isolated. If $x(t)$ is a periodic solution, then so is $\alpha x(t)$ for any constant $\alpha \neq 0$. Hence $x(t)$ is surrounded by a 'family' of closed orbits. Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions. Any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

Limit cycles are only possible in systems with dissipation. System that conserve energy do not have isolated closed trajectories.

   (a) (5 points) Accept the assignment in GitHub Classroom, launch the codespace, open the template of the notebook for the assignment, `midpoint.ipynb`.

   (b) (10 points) Use Markdown to fill in the blanks in the introductory part of the notebook.

(c) (5 points) Rewrite van der Pol equation as a system of first order differential equations. Show your work in the space below.

(d) (30 points) Consider the IVP Eq. (1). Consider separately two cases: week nonlinearity, $\varepsilon = 0.1$, with the initial conditions $x(0) = 1.5$, $\dot{x}(0) = 0$, and strong nonlinearity, $\varepsilon = 10$, with the initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$. Solve Eq. (1) using Julia's `OrdinaryDiffEqTsit5` package. Consider the range of the independent variable t that is sufficiently long so that the trajectories settles on the limit cycles or spirals away from it. (The ranges are different for week and strong nonlinearity cases.) On two different figures plot the phase trajectories $\dot{x}(x)$. (You must have sufficient number of data points so that the trajectories appear as smooth curves.) Provide the grid, title, axes labels for each of your graphs.

(e) (15 points) Are the limit cycles of the van der Pol equation stable or unstable? Conduct numerical experiments for 3 different initial conditions for each of weak and strong nonlinearity case. Present your results and describe your reasoning.

(f) (5 points) Clean the cells of your jupyter notebook(s), save and close the notebook(s). Delete unneeded notebooks if you created any (e.g. `Untitled.ipynb`). Commit all your changes to the project and push them to the assignment's GitHub repository.

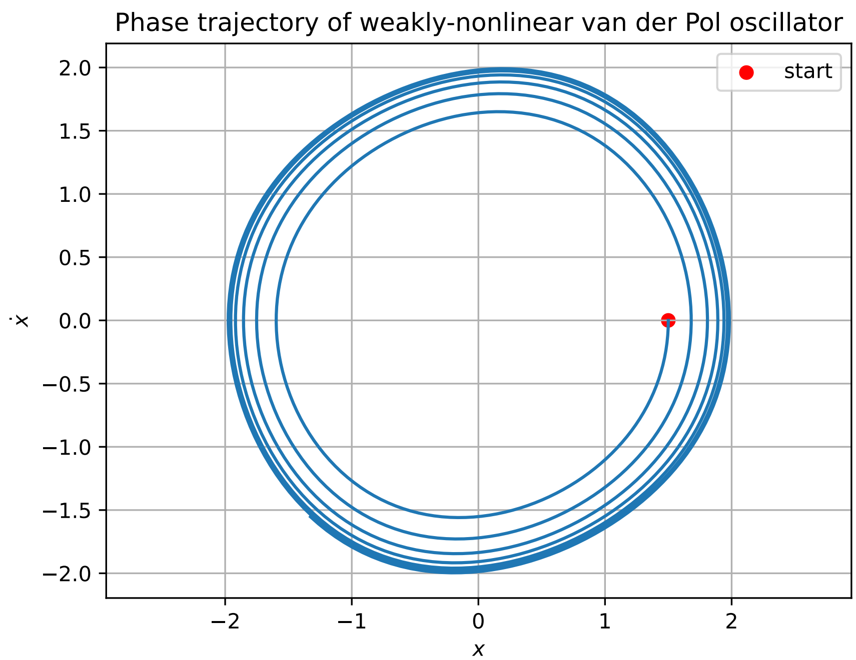

Figure 1: Typical phase trajectory of weakly-nonlinear van der Pol oscillator (colors online).

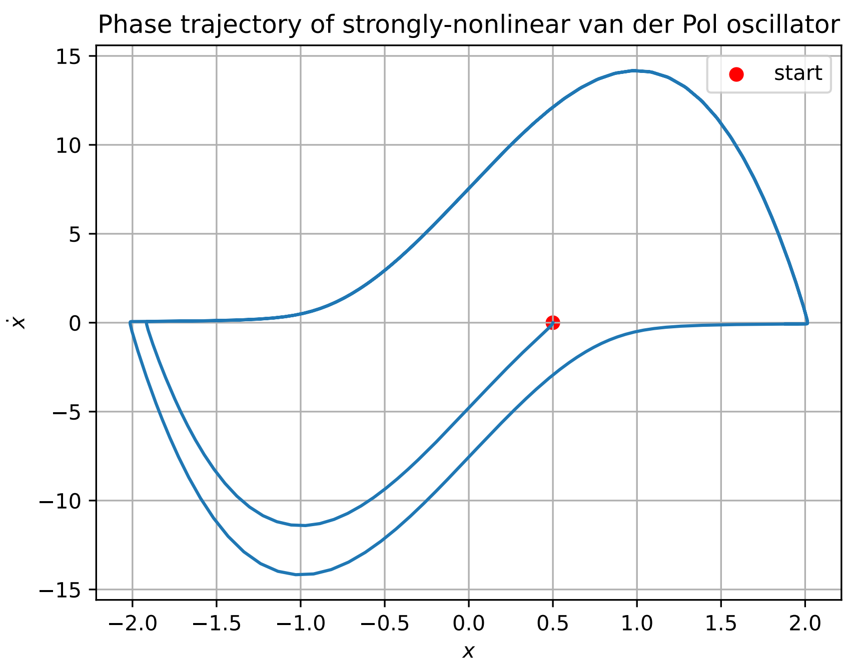

Figure 2: Typical phase trajectory of strongly-nonlinear van der Pol oscillator (colors online). Notice the different scales on x and $\dot{x}$ axes.