GLIDER FLIGHT

Fall semester 2024

https://www.phys.uconn.edu/~rozman/Courses/P2200_24F/

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1 Experimental facts

Consider a glider which motion is restricted, just to simplify the discussion, to a vertical plane (see Fig 1).

There are three forces acting on the glider: L, the lift, W, the force of gravity, and D, the drag. The drag, D, is anti-parallel to the instantaneous direction of the velocity (tangential direction to the trajectory). The lift, L, created by the airflow around the wings is perpendicular to the direction of the velocity.

Both lift and drag are proportional to a surface area of the wings, S, and to the dynamic pressure, $1/2\rho V^2$, where ρ is the density of air, and V the forward velocity of the aircraft. Both forces can be expressed in terms of coefficients of lift and drag, C_L and C_D, respectively:

$$L = \frac{1}{2}\rho V^2 S C_L, \tag{1}$$

$$D = \frac{1}{2}\rho V^2 S C_D. \tag{2}$$

The ratio of lift to drag, L/D, is called the *aerodynamic efficiency* of the aircraft.

$$R \equiv \frac{L}{D} = \frac{C_L}{C_D}. \tag{3}$$

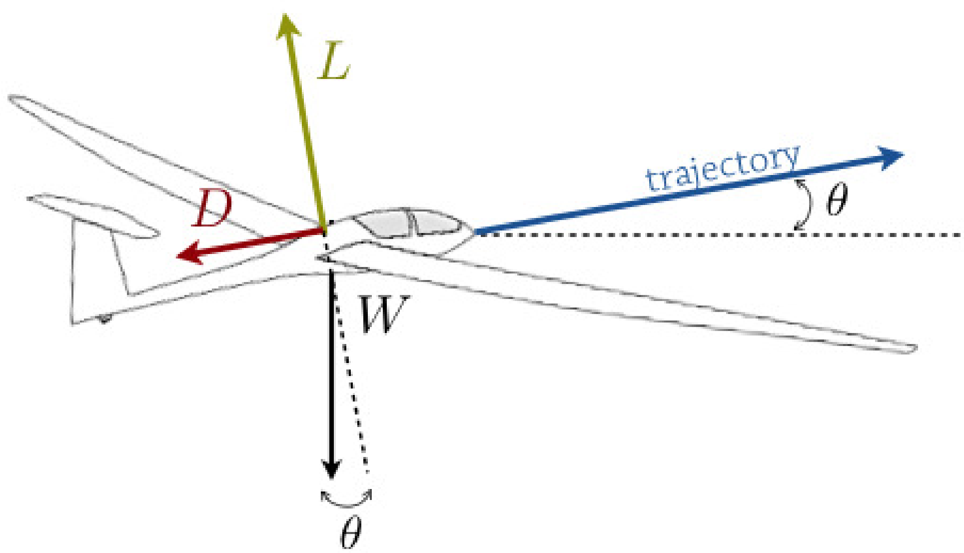

Figure 1: Forces acting on the glider: L, the lift, W, the force of gravity, $W = mg$, and D, the drag. θ is the angle between the instantaneous direction of the velocity (tangential direction to the trajectory) and the horizontal line.

2 Equations of motion

Newton's second law of motion applied to the motion tangential to the trajectory

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -mg\sin\theta - D, \tag{4}$$

where V is the speed of the glider, m is its mass, g is acceleration of gravity, D is the drag force given by Eq. (2).

The minus sign in front of the first term in the right hand side of Eq. (4) corresponds to the 'up' direction as positive for θ: when θ is negative, the nose is pointing down and the plane accelerates due to gravity. When $\theta > 0$, the plane must fight against gravity and its acceleration due to gravity is negative.

In the normal direction, we have centripetal force, $m\frac{V^2}{r}$, where r is the instantaneous radius of curvature. After noticing that that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{V}{r},$$

the centripetal force can be expressed as $mV\frac{\mathrm{d}\theta}{\mathrm{d}t}$, giving

$$mV\frac{\mathrm{d}\theta}{\mathrm{d}t} = -mg\cos\theta + L, \tag{5}$$

where L is the lift force given by Eq. (1).

Thus, we have a system of two first order nonlinear ordinary differential equations for $V(t)$ and $\theta(t)$.

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -mg\sin\theta - \frac{1}{2}\rho V^2 C_D S, \tag{6}$$

$$mV\frac{d\theta}{dt} = -mg\cos\theta + \frac{1}{2}\rho V^2 C_L S. \tag{7}$$

Eqs. (6), (7) were derived first by Joukowski[1] in 1891 and by Lanchester[2] in 1908 (see [3, p. 149] for a brief history and attribution).

3 Glider trajectory

The spatial coordinates of the glider with respect to a 'ground' reference frame, $X(t)$ and $Y(t)$, could be determined by integration, once $V(t)$ and $\theta(t)$ are found:

$$X(t) = X(0) + \int_0^t V(t')\cos(\theta(t'))\,\mathrm{d}t', \tag{8}$$

$$Y(t) = Y(0) + \int_0^t V(t')\sin(\theta(t'))\,\mathrm{d}t'. \tag{9}$$

However, for numerical solution it is more convenient to re-write Eqs. (8), (9) in the following differential form:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V\cos\theta, \tag{10}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = V\sin\theta. \tag{11}$$

The equations (6), (7) and (10), (11) for a system of four first order nonlinear ordinary differential equations for unknowns $V(t)$, $\theta(t)$, $X(t)$, and $Y(t)$.

4 Scaling the equations

The equations (6), (7), (10), (11) contain five parameters: the mass of the glider, m, the density of the air, ρ, the area of the wings, S, and the coefficients of lift and drag, C_L and C_D, as well as one constant - acceleration of gravity g. It is often useful to reduce equations describing a physical system to dimensionless form, both for physical insight and for numerical convenience (i.e., to avoid dealing with very large or very small numbers in the computer). To do this for the equations of glider motion, we introduce dimensionless velocity, time, and length variables.

To introduce the characteristic velocity, let's consider horizontal motion of the glider:

$$\theta = 0, \quad V = v_t = \text{const.} \tag{12}$$

From Eq. (7) we conclude that such motion is possible if the force of gravity is balanced by the lift force:

$$mg = \frac{1}{2}\rho v_t^2 C_L S. \tag{13}$$

The relation Eq. (13) introduces the characteristic velocity, v_t.

$$v_t \equiv \sqrt{\frac{mg}{\frac{1}{2}\rho C_L S}}. \tag{14}$$

For the reference,

$$m = \frac{1}{2g}\rho v_t^2 C_L S. \tag{15}$$

We are going to measure the speed of the glider in units of v_t, i.e. we introduce a dimensionless speed variable, v:

$$v \equiv \frac{V}{v_t}. \tag{16}$$

The characteristic acceleration in the glider problem is acceleration of gravity, g. We can combine the characteristic acceleration with the characteristic velocity, v_t, to get a characteristic variable, t_c with the dimension of time:

$$t_c = \frac{v_t}{g}. \tag{17}$$

t_c is the time for a free falling body starting from rest to gain the speed v_t.

We are going to measure the time in units of t_c, i.e. we introduce new dimensionless time variable, τ:

$$\tau \equiv \frac{t}{t_c}, \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{t_c}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{g}{v_t}\frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{19}$$

Finally, we introduce the characteristic length as the product of v_t and t_c:

$$l_c = t_c v_t = \frac{v_t^2}{g} = \frac{m}{\frac{1}{2}\rho C_L S}. \tag{20}$$

l_c is equal double the height for a free falling body starting from rest to gain the speed v_t.

We'll measure the X and Y coordinates of the glider in units of l_c:

$$x = \frac{X}{l_c}, \quad y = \frac{Y}{l_c}. \tag{21}$$

Substituting Eqs. (13), (15)–(17), and (19) into Eqs. (6)–(7) and (10)–(11), we obtain:

$$\frac{1}{2g}\rho v_t^2 C_L S v_t \frac{g}{v_t}\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{1}{2}\rho v_t^2 C_L S \sin\theta - \frac{1}{2}\rho v_t^2 v^2 C_D S, \tag{22}$$

$$\frac{1}{2g}\rho v_t^2 C_L S v v_t \frac{g}{v_t}\frac{d\theta}{d\tau} = -\frac{1}{2}\rho v_t^2 C_L S \cos\theta + \frac{1}{2}\rho v_t^2 v^2 C_L S, \tag{23}$$

$$t_c v_t \frac{1}{t_c}\frac{\mathrm{d}x}{\mathrm{d}\tau} = v_t v \cos\theta, \tag{24}$$

$$t_c v_t \frac{1}{t_c}\frac{\mathrm{d}y}{\mathrm{d}\tau} = v_t v \sin\theta. \tag{25}$$

Canceling common factors, arrive to the following system of ODEs:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\sin\theta - \frac{v^2}{R}, \tag{26}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{\cos\theta}{v} + v, \tag{27}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = v\cos\theta, \tag{28}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = v\sin\theta. \tag{29}$$

This system of equations contains a single parameter – the aerodynamic efficiency R, Eq. (3).

5 Steady state flight

One way we can get a better handle on exactly how the behavior of the solutions depends on the aerodynamic efficiency of the aircraft, R, is to examine what happens to solutions near the constant solution. We notice that whenever $\dot{\theta} = 0$ and $\dot{v} = 0$, we must have a constant solution:

$$\sin\theta = -\frac{v^2}{R}, \tag{30}$$
$$\cos\theta = v^2. \tag{31}$$

Dividing Eq. (30) by Eq. (31), we obtain:

$$\theta = -\arctan\left(\frac{1}{R^2}\right). \tag{32}$$

Squaring Eqs. (30) and (31) and adding them, we get:

$$v = \frac{1}{\sqrt[4]{1 + \frac{1}{R^2}}}. \tag{33}$$

6 Numerical calculations

See Fig 2 for several typical trajectories.

References

[1] Nikolai Joukowski. "On Soaring of Birds". In: *Trudy Otdeleniia Fizicheskih Nauk, Obshchestvo Liubitelei Estestvoznaniia* 4.2 (1891). In Russian, pp. 29–43.
[2] Frederick Lanchester. *Aerodonetics*. London, 1908.
[3] Theodore Von Karman. *Aerodynamics*. McGraw-Hill Education, 1963.

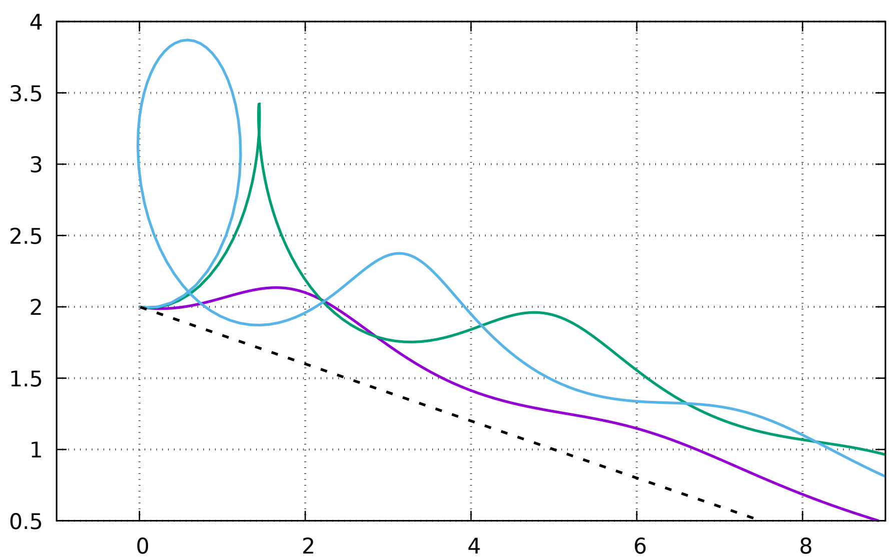

Figure 2: Typical trajectories of a glider (color online).