THE STRUCTURE OF WHITE DWARF STARS

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1 Introduction

White dwarfs are the final evolutionary state of stars whose mass is not high enough to become a neutron star or a black hole. After the hydrogen–fusing (H $\to$ He) lifetime of a star ends, such a star fuses helium to carbon and oxygen, (He $\to$ C, O). If a star has insufficient mass to generate the core temperatures required to further fuse carbon and oxygen, an inert mass of carbon and oxygen will build up at its center. After shedding its outer layers, the star will leave behind the core, which is the white dwarf.

The material in a white dwarf no longer undergoes fusion reactions, so the star is not supported by the heat generated by fusion against gravitational collapse. It is supported only by *electron degeneracy* pressure, causing the star to be extremely dense.

This notes present a theoretical description of the internal structure of white dwarfs and determine the dependence of the radius of a white dwarf stars versus its mass. Both the radius and the mass can be determined from the results of astronomical observations (see Fig. 1) and thus the predictions of the theory can be verified.

2 The equations of the mechanical equilibrium

If the star is in mechanical equilibrium, the gravitational force at each point inside is balanced by the force due to the spatial variation of the pressure P. The gravitational force acting on a unit volume of matter at a radius r is

$$F_{\text{grav}} = -G\,\frac{m(r)\,\rho(r)}{r^2}, \tag{1}$$

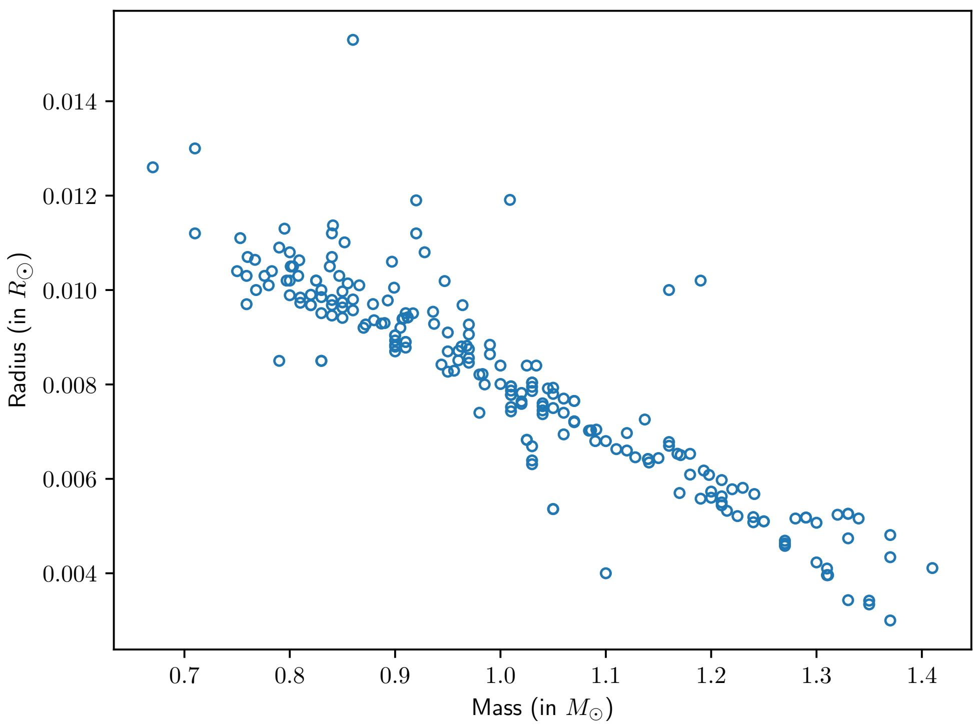

Figure 1: Observational support for the white dwarf mass-radius relation [1]. In the axis labels $M_\odot$ and $R_\odot$ are the mass and the radius of the Sun.

where G is the gravitational constant, $\rho(r)$ is the mass density of the star, and $m(r)$ is the mass of the star interior to the radius r:

$$m(r) = \int \rho \, \mathrm{d}V = 4\pi \int_0^r \rho(r') r'^2 \, \mathrm{d}r', \tag{2}$$

where we used the volume of spherical shell or radius r and thickness $\mathrm{d}r$:

$$\mathrm{d}V = 4\pi r^2 \mathrm{d}r.$$

A differential relation between the mass, $m(r)$, and the density, $\rho(r)$, can be obtained by differentiating the Eq. (2) with respect to r:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r). \tag{3}$$

The radial component of the force per unit volume of matter due to the changing pressure is as following:

$$F_{\mathrm{r}} = \frac{\mathrm{d}P}{\mathrm{d}r}. \tag{4}$$

When the star is in equilibrium, we thus have:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\,\frac{m(r)\,\rho(r)}{r^2}. \tag{5}$$

The description of mechanical equilibrium is completed by specifying *the equation of state*, a relation that gives the pressure, $P = P(\rho)$, which is required to maintain the matter at a given density, ρ. Using the identity

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{6}$$

Eq. (5) can be written as

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)^{-1} \frac{G\,m(r)}{r^2}\,\rho(r). \tag{7}$$

Equations (3) and (7) are two coupled first-order differential equations for $\rho(r)$ and $m(r)$ that determine the structure of the star for a given equation of state. The values of the dependent variables at $r = 0$ are $\rho(0) = \rho_c$, the (unknown) central density, and $m(0) = 0$. Integration outward in r then gives the density and mass profiles. The radius of the star, R, is being determined by the point at which $\rho = 0$. The total mass of the star is then $M = m(R)$. Since both R and M depend upon ρ_c, variation of this parameter allows to determine the mass-radius relation for white dwarf stars $R(M)$.

3 The equation of state

To be able to solve Equations (3) and (7), we need the equation of state for a white dwarf. We assume that the matter consists of a single kind large nuclei (e.g. oxygen) and their electrons. The nuclei, being heavy, contribute nearly all of the mass but make almost no contribution to the pressure since they hardly move at all. The electrons, however, contribute virtually all of the pressure but essentially none of the mass. We will be interested in densities far greater than that of ordinary matter, where the electrons are no longer bound to individual nuclei, but rather move freely through the material. A good model is then a free gas of electrons at zero temperature, treated with relativistic kinematics.

The quantum mechanical theory, which also takes into account the relativistic expression for electrons' kinetic energy[2], gives the following result:

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = Y_e \frac{m_e c^2}{M_p} \gamma(\rho), \tag{8}$$

where M_p is the mass of the proton, m_e is the mass of of the electron, Y_e is the number of electrons per nucleon, c is the speed of light, and dimensionless function $\gamma(\rho)$ is

$$\gamma(\rho) = \frac{\left(\frac{\rho}{\rho_0}\right)^{2/3}}{3\sqrt{1+\left(\frac{\rho}{\rho_0}\right)^{2/3}}}. \tag{9}$$

Here

$$\rho_0 = \frac{M_p m_e^3 c^3}{3\pi^2 \hbar^3 Y_e}. \tag{10}$$

Using Eq.(8) and Eq.(7) we get the following differential equation governing the evolution of $\rho(r)$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\left(\frac{M_p}{m_e c^2 Y_e}\right) \frac{G m(r)}{\gamma(\rho) r^2} \rho(r). \tag{11}$$

To avoid numerical difficulties in calculating the right hand side of Eq. (11) for small values of r, notice that for sufficiently small r

$$m(r) \approx \frac{4}{3}\pi r^3 \rho_c. \tag{12}$$

Hence, for small r Eq. (11) can be written in the following form:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{4}{3}\pi\left(\frac{M_p}{m_e c^2 Y_e}\right) \frac{G r}{\gamma(\rho_c)} \rho_c^2, \tag{13}$$

which avoids diverging factor $1/r^2$.

4 Scaling the Differential Equations

It is always useful to reduce equations describing a physical system to dimensionless form, both for physical insight and for numerical convenience. To do this for the equations of white dwarf structure, we introduce dimensionless radius, density, and mass variables:

$$r = R_0 \bar{r}, \quad \rho = \rho_0 \bar{\rho}, \quad m = M_0 \bar{m} \tag{14}$$

with the radius and mass scales, R_0 and M_0 to be determined for convenience.

Substituting Eq. (14) into Eqs. (3), (11) yields

$$\frac{d\bar{m}}{d\bar{r}} = \left(\frac{4\pi R_0^3 \rho_0}{M_0}\right) \bar{r}^2 \bar{\rho} \tag{15}$$

and

$$\frac{d\bar{\rho}}{d\bar{r}} = -\left(\frac{G M_p M_0}{m_e c^2 Y_e R_0}\right) \frac{\bar{m}\bar{\rho}}{\gamma(\bar{\rho})\bar{r}^2}. \tag{16}$$

If we now choose M_0 and R_0 so that the coefficients in parentheses in these two equations are ones, we find

$$R_0 = \left(\frac{m_e c^2 Y_e}{4\pi \rho_0 G M_p}\right)^{\frac{1}{2}} = 7.71 \times 10^3\, Y_e \text{ km}, \tag{17}$$

and

$$M_0 = 4\pi R_0^3 \rho_0 = 5.66 \times 10^{30}\, Y_e^2 \text{ kg}. \tag{18}$$

If we consider a white dwarf star consisting of ^{12}C, a chemical element with 6 protons, six neutrons, and six electrons, then $Y_e = \frac{1}{2}$ and $M_0 = 0.71 \times M_\odot$ and $R_0 = 0.006 \times R_\odot$, where $M_\odot$ and $R_\odot$ are the mass and the radius of the Sun.

The dimensionless differential equations are

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{r}} = \bar{r}^2 \bar{\rho}, \tag{19}$$

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma(\bar{\rho})\bar{r}^2}. \tag{20}$$

Here γ, defined by Eq. (9), is

$$\gamma(\bar{\rho}) = \frac{\bar{\rho}^{\,2/3}}{3\sqrt{1+\bar{\rho}^{\,2/3}}}. \tag{21}$$

This pair of equations is then integrated from $\bar{r} = 0$, $\bar{\rho} = \bar{\rho}_c$, $\bar{m} = 0$ to the value of $\bar{r}$ at which $\bar{\rho} = 0$, which defines the dimensionless radius of the star $\bar{R}$, and the dimensionless mass of the star is then $\bar{M} = \bar{m}(\bar{R})$.

At the initial stage of numerical integration when $\bar{r} \ll 1$, from Eq. (19),

$$\bar{m} \approx \frac{1}{3}\bar{r}^3 \bar{\rho}_c. \tag{22}$$

Thus, for small $\bar{r}$ Eq. (20) can be rewritten in the following form that avoids the diverging $\bar{r}^2$ factor in the denominator:

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{r}} = -\frac{\bar{r}\bar{\rho}_c^2}{3\gamma(\bar{\rho}_c)}. \tag{23}$$

5 Results of calculations. Chandrasekhar limit.

The results of numerical integration of Eqs. (19) and (20) are presented in Fig. 2. There is a strikingly good agreement between the theory and the observations.

The theory, in agreement with the observations, predicts that more massive white dwarfs have smaller radii. Therefore, there is a critical mass, M_c for which the predicted radius is zero. This limited mass is called *Chandrasekhar limit*. From the results presented in Fig. 2,

$$M_c \approx 1.44 M_{\odot}. \tag{24}$$

White dwarfs resist gravitational collapse through electron degeneracy pressure. The Chandrasekhar limit is the mass above which electron degeneracy pressure in the star's core is insufficient to balance the star's own gravitational self-attraction. Consequently, a star with a mass greater than the limit is subject to further gravitational collapse, evolving into a neutron star or black hole.

References

[1] F. Ochsenbein, P. Bauer, and J. Marcout. “The VizieR database of astronomical catalogues”. In: *Astronomy and Astrophysics Supplement* 143 (2000), pp. 23–32.

[2] S.E. Koonin. *Computational Physics*. Benjamin/Cummings Publishing Company, 1986.

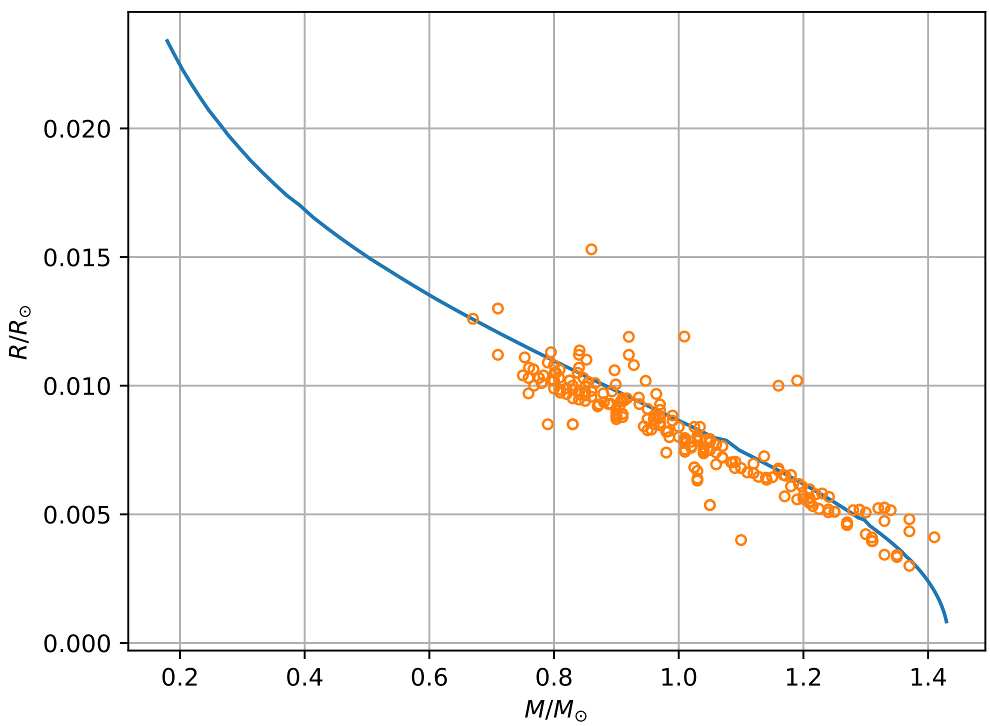

Figure 2: White dwarf mass-radius relation: comparison between observational data (scatterplot, see also Fig. 1) and the theory (solid line). In the axis labels $M_\odot$ and $R_\odot$ are the mass and the radius of the Sun.