SIMPSON'S INTEGRATION RULE

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https://www.phys.uconn.edu/~rozman/Courses/P2200_23F/

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Simpson's integration formula can be obtained by applying *Richardson's extrapolation* to the trapezoidal integration formula. We begin with the trapezoidal rule with the integration step h:

$$\int_a^b f(x)\,\mathrm{d}x = \frac{h}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-1}) + f(x_n)\right] + \alpha h^2 + O(h^p), \quad (1)$$

where

$$h = \frac{b-a}{n-1}, \quad (2)$$

$$x_i = a + (i-1)\,h, \quad i = 1, \ldots, n, \quad (3)$$

αh^2 is the leading discretization error term of the algorithm, the coefficient α is unknown. The term $O(h^p)$ indicates the higher order error terms, $p > 2$. We assume that the number of nodes, n, is an odd number.

Next, we write down the trapezoidal rule with step $2h$. It will involve only the nodes at x_1, x_3, x_5, ..., x_{n-2}, x_n:

$$\int_a^b f(x)\,\mathrm{d}x = h\left[f(x_1) + 2f(x_3) + 2f(x_5) + \ldots + 2f(x_{n-2}) + f(x_n)\right] + \alpha(2h)^2 + O(h^p). \quad (4)$$

The leading error term in Eq. (4) is 4 times the error in Eq. (1), since we doubled the step size. So we multiply both sides of Eq. (1) by 4,

$$4\int_a^b f(x)\,\mathrm{d}x = h\left[2f(x_1) + 4f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + 2f(x_n)\right] + 4\alpha h^2 + O(h^p), \quad (5)$$

and subtract Eq. (4) from Eq. (5), intending to cancel out the leading error term. The result that we get is the Simpson's formula:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\right] + O(h^p). \quad (6)$$

We can repeat this process to obtain the higher order integration rules called *Newton-Cotes integration rules*.