RESTRICTED THREE BODY PROBLEM

Fall 2023 semester

https://www.phys.uconn.edu/~rozman/Courses/P2200_23F/

Last modified: December 7, 2023

1 Introduction

An isolated system consisting of two point masses exerting forces on one another — which is usually referred to as a *two-body problem* — can always be solved by converting it into an equivalent one-body problem. In particular, we can exactly solve a dynamical system containing two gravitationally interacting point masses. What about a system containing *three* gravitationally interacting point masses? Despite hundreds of years of research, no exact solution of this famous problem — which is generally known as the *three-body problem* — has ever been found. It is, however, possible to make some progress by restricting the problem's scope.

2 The circular restricted three-body problem

Let's consider a mechanical system consisting of three point masses, M_1 , M_2 , and m. Suppose, that mass m is much smaller than the other two so that it has a negligible effect on their motion. Suppose, further, that the first two masses, M_1 and M_2 , execute circular orbits about their common center of mass. This simplified problem is known as the *circular restricted three-body problem*.

Let's assume, to simplify the presentation of the final calculations, that mass m moves in the plane of the orbital motion of masses M_1 and M_2 .

Let ω be the orbital angular velocity of masses M_1 and M_2 on the circular orbit. We can find ω by equating F_{cp} , the centripetal force acting upon the mass $\mu = \frac{M_1M_2}{M_1+M_2}$ (the equivalent



Figure 1: The circular restricted three-body problem.

one-body problem), and F_g , the force of gravitational attraction between masses M_1 and M_2 :

$$F_{cp} = \frac{M_1 M_2}{M_1 + M_2} \frac{v^2}{R}, \qquad F_g = G \frac{M_1 M_2}{R^2}, \tag{1}$$

where G is the gravitational constant, v is the constant linear velocity of mass μ . From Eq. (1)

$$v^2 = G \,\frac{M_1 + M_2}{R}.$$
 (2)

The period of orbital motion on a circular orbit, *T*, is

$$T = \frac{2\pi R}{v},\tag{3}$$

thus,

$$\omega \equiv \frac{2\pi}{T} = \frac{v}{R}, \qquad \omega^2 = \frac{v^2}{R^2}.$$
(4)

Substituting Eq. (2) into Eq. (4), we arrive at the following expression.

$$\omega^2 = G \frac{M_1 + M_2}{R^3}.$$
 (5)

Let us define a Cartesian coordinate system (ξ, η, ζ) in an inertial reference frame whose origin coincides with the center of mass, *C*, of the two orbiting masses, M_1 and M_2 . Let the orbital plane of these masses coincide with the ξ - η plane, and let them both lie on the ξ -axis at time t = 0 — see Figure 1. Suppose that *R* is the constant distance between the two orbiting masses, r_1 the constant distance between mass M_1 and the origin, and r_2 the constant distance between mass M_2 and the origin. Let the third mass have position vector $\vec{r} = (\xi, \eta, 0)$. The Cartesian components of the equation of motion of this mass are thus

$$\ddot{\xi} = -GM_1 \frac{(\xi - \xi_1)}{\rho_1^3} - GM_2 \frac{(\xi - \xi_2)}{\rho_2^3},$$
(6)

$$\ddot{\eta} = -GM_1 \frac{(\eta - \eta_1)}{\rho_1^3} - GM_2 \frac{(\eta - \eta_2)}{\rho_2^3},\tag{7}$$

where (ξ_1, η_1) and (ξ_2, η_2) are the coordinates of masses M_1 and M_2 respectively, and

$$\rho_1^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2, \tag{8}$$

$$\rho_2^2 = (\xi - \xi_2)^2 + (\eta - \eta_2)^2.$$
(9)

 ρ_1 and ρ_2 are the distances between mass *m* and masses M_1 and M_2 respectively.

3 Co-rotating reference frame

Let us transform to a non-inertial frame of reference rotating with angular velocity ω about an axis normal to the orbital plane of masses M_1 and M_2 , and passing through their center of mass. The masses M_1 and M_2 are *stationary* in this new reference frame. Let us define a Cartesian coordinate system (X, Y) in the rotating frame of reference which is such that masses M_1 and M_2 always lie on the X-axis. Let the position vector of mass m be $\vec{r} = (x, y)$ see Figure 2.

The masses M_1 and M_2 have the fixed position vectors

$$\vec{r}_1 = (-\alpha R, 0, 0)$$
 $\vec{r}_2 = ((1 - \alpha)R, 0, 0)$ (10)

in our new coordinate system. Indeed, by the definition of the center of mass,

$$r_1 M_1 = r_2 M_2. (11)$$

on the other hand,

$$r_1 + r_2 = R. (12)$$

Solving Eqs. (11) and (12), we obtain,

$$r_1 = \frac{M_2}{M_1 + M_2} R, \qquad r_2 = \frac{M_1}{M_1 + M_2} R = \left(1 - \frac{M_2}{M_1 + M_2}\right) R,$$
 (13)

i.e. in Eq. (10)

$$\alpha = \frac{M_2}{M_1 + M_2}.\tag{14}$$



Figure 2: The co-rotating frame.

The equation of motion of mass *m* in the rotating reference frame are obtained by including into Eqs. (6), (7) two additional forces — Coriolis force \vec{F}_{cor} and centrifugal force \vec{F}_{cf} :

$$\vec{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 \vec{r}, \tag{15}$$

$$\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r}} = 2m\omega \left(-\hat{x}\dot{y} + \hat{y}\dot{x}\right) \tag{16}$$

$$\ddot{\vec{r}} = -GM_1 \frac{(\vec{r} - \vec{r_1})}{\rho_1^3} - GM_2 \frac{(\vec{r} - \vec{r_2})}{\rho_2^3} + \omega^2 \vec{r} - 2 \vec{\omega} \times \dot{\vec{r}},$$
(17)

where $\vec{\omega} = (0, 0, \omega)$, and

$$\rho_1^2 = (x + \alpha R)^2 + y^2, \tag{18}$$

$$\rho_2^2 = (x - (1 - \alpha)R)^2 + y^2.$$
(19)

The components of Eq. (17) reduce to

$$\ddot{x} = -\frac{GM_1(x+\alpha R)}{\rho_1^3} - \frac{GM_2(x-(1-\alpha)R)}{\rho_2^3} + \omega^2 x + 2\omega \dot{y},$$
(20)

$$\ddot{y} = -\frac{GM_1 y}{\rho_1^3} - \frac{GM_2 y}{\rho_2^3} + \omega^2 y - 2 \omega \dot{x}.$$
(21)

4 Jacobi integral

Eqs. (20), (21) can be rewritten as following.

$$\ddot{x} - 2\omega \dot{y} = -\frac{\partial U}{\partial x}, \qquad (22)$$

$$\ddot{y} + 2\omega \dot{x} = -\frac{\partial U}{\partial y}.$$
(23)

where

$$U(x,y) = -\frac{GM_1}{\rho_1} - \frac{GM_2}{\rho_2} - \frac{\omega^2}{2}(x^2 + y^2)$$
(24)

is the sum of the gravitational and centrifugal potentials.

Now, it follows from Eqs. (22)-(23) that

$$\ddot{x}\dot{x} - 2\omega\dot{x}\dot{y} = -\dot{x}\frac{\partial U}{\partial x},$$
(25)

$$\ddot{y}\dot{y} + 2\omega \dot{x}\dot{y} = -\dot{y}\frac{\partial U}{\partial y}.$$
(26)

Summing the above equations, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\left(\dot{x}^{2}+\dot{y}^{2}\right)+U(x,y)\right]=0.$$
(27)

In other words,

$$C = -2U - v^2 \tag{28}$$

is a *constant of the motion*, where $v^2 = \dot{x}^2 + \dot{y}^2$. *C* is called the *Jacobi integral*. The trajectory mass *m* is restricted to regions in which

$$-2 U \ge C, \tag{29}$$

since v^2 is a positive definite quantity.

5 Dimensionless form of the equations

No analytic solutions of Eqs. (20)–(21) are known. Our goal is to solve them numerically. As the first step, we convert them to a dimensionless form.

Circular restricted three body problem has two natural scales: the distance, R, between masses M_1 and M_2 , and the characteristic time of their orbital motion $1/\omega$. Let us introduce

dimensionless variables by measuring the coordinates x and y in units of R, thus introducing new unknowns u and v as following,

$$u \equiv \frac{x}{R}, \qquad v \equiv \frac{y}{R}, \tag{30}$$

Let us measure time t in units of $1/\omega$, introducing dimensionless variable τ ,

$$\tau \equiv \omega t. \tag{31}$$

"Old" derivatives with respect to time are going to have the following forms:

$$\dot{x} \equiv \frac{dx}{dt} = \frac{d(uR)}{d(\tau/\omega)} = \omega R \frac{du}{d\tau},$$
(32)

$$\ddot{x} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d}{dt} \left(\omega R \frac{du}{d\tau} \right) = \omega R \frac{d}{d\tau/\omega} \left(\frac{du}{d\tau} \right) = \omega^2 R \frac{d^2 u}{d\tau^2}.$$
(33)

Similarly,

$$\dot{y} = \omega R \frac{dv}{d\tau} \tag{34}$$

$$\ddot{y} = \omega^2 R \frac{d^2 v}{d\tau^2} \tag{35}$$

Substituting Eqs. (32)– (35) into Eqs. (20), (21), we get:

$$\omega^{2}R\frac{d^{2}u}{d\tau^{2}} = -\frac{GM_{1}R(u+\alpha)}{\rho_{1}^{3}} - \frac{GM_{2}R(u-1+\alpha)}{\rho_{2}^{3}} + \omega^{2}Ru + 2\omega^{2}R\frac{dv}{d\tau},$$
 (36)

$$\omega^2 R \frac{d^2 v}{d\tau^2} = -\frac{GM_1 R v}{\rho_1^3} - \frac{GM_2 R v}{\rho_2^3} + \omega^2 R v - 2 \omega^2 R \frac{du}{d\tau}.$$
 (37)

Here ρ_1 and ρ_2 expressed via dimensionless parameters are as following:

$$\rho_1 = R\left((u+\alpha)^2 + v^2\right)^{\frac{1}{2}} = Rd_1,$$
(38)

$$\rho_2 = R\left((u-1+\alpha)^2 + v^2\right)^{\frac{1}{2}} = Rd_2, \tag{39}$$

where

$$d_{1} \equiv \left((u+\alpha)^{2} + v^{2} \right)^{\frac{1}{2}},$$
(40)

$$d_2 \equiv \left((u - 1 + \alpha)^2 + v^2 \right)^{\frac{1}{2}}.$$
 (41)

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Dividing each term in Eqs. (36)– (37) by $\omega^2 R$, we arrive at the following equations:

$$\frac{d^2 u}{d\tau^2} = -\frac{GM_1}{\omega^2 R^3} \frac{(u+\alpha)}{d_1^3} - \frac{GM_2}{\omega^2 R^3} \frac{(u-1+\alpha)}{d_2^3} + u + 2\frac{dv}{d\tau},$$
(42)

$$\frac{d^2v}{d\tau^2} = -\frac{GM_1}{\omega^2 R^3} \frac{v}{d_1^3} - \frac{GM_2}{\omega^2 R^3} \frac{v}{d_2^3} + v - 2\frac{du}{d\tau}.$$
(43)

Noticing that

$$\frac{GM_1}{\omega^2 R^3} = \frac{M_1}{M_1 + M_2} \equiv 1 - \alpha$$
(44)

and

$$\frac{GM_2}{\omega^2 R^3} = \frac{M_2}{M_1 + M_2} \equiv \alpha \tag{45}$$

we arrive at the following equations.

$$\frac{d^2 u}{d\tau^2} = -(1-\alpha)\frac{(u+\alpha)}{d_1^3} - \alpha \frac{(u-1+\alpha)}{d_2^3} + u + 2\frac{dv}{d\tau},$$
(46)

$$\frac{d^2v}{d\tau^2} = -(1-\alpha)\frac{v}{d_1^3} - \alpha\frac{v}{d_2^3} + v - 2\frac{du}{d\tau}.$$
(47)

Equations (46)- (47) can be rewritten in a compact form

$$\ddot{u} = -\frac{\partial U}{\partial v} + 2\dot{v}, \qquad (48)$$

$$\ddot{v} = -\frac{\partial U}{\partial v} - 2\dot{u}, \qquad (49)$$

where

$$U(u,v) = -\frac{1-\alpha}{d_1} - \frac{\alpha}{d_2} - \frac{1}{2}\left(u^2 + v^2\right)$$
(50)

is the dimensionless version of Eq. (24).

Equations (46)- (47) are dimensionless and contain a single parameter, α . Some of the results of their numerical solution are presented in Figs. 3 and 4. A fragment of the code used for calculations is presented in the Appendix A.



Figure 3: Arenstorf periodic orbits for $\alpha = 0.012277471$ (Earth–Moon system) and initial conditions x(0) = 0.994, y(0) = 0, $\dot{x}(0) = 0$; left subfigure: $\dot{y}(0) = -2.0317326295573368357302057924$, right subfigure: $\dot{y}(0) = -2.00158510637908252240537862224$,



Figure 4: Chaotic orbit: $\alpha = 0.5$, x(0) = 1, y(0) = 0, $\dot{x}(0) = 0$, $\dot{y}(0) = 0$.

Appendix A

A fragment of a Julia code to solve the restricted three-body problem using OrdinaryDiffEq package.

```
function arensdorf!(dudt, u, p, t)
    x, y, dxdt, dydt = u
    mu = p
    mu1 = 1.0 - mu
    d(r) = ((x + r)^2 + y^2)^(3/2)
    d1 = d(mu)
    d2 = d(-mu1)
    dudt[1] = u[3]
    dudt[2] = u[4]
    dudt[3] = x + 2*dydt - mu1*(x + mu)/d1 - mu*(x - mu1)/d2
    dudt[4] = y - 2*dxdt - mu1*y/d1 - mu*y/d2
end
```

References

- [1] Richard Fitzpatrick. "Newtonian Dynamics". 2011. URL: https://farside.ph.utexas. edu/teaching/336k/336k.html.
- [2] Richard F. Arenstorf. "Periodic Solutions of the Restricted Three Body Problem Representing Analytic Continuations of Keplerian Elliptic Motion". In: American Journal of Mathematics 85 (1963), pp. 27–35.