MONTE CARLO CALCULATIONS OF EULER-MASCHERONI CONSTANT

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The Euler–Mascheroni constant is defined as the limiting difference between harmonic series and the natural logarithm.

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right). \tag{1}$$

An integral representation

One of many known integral representations of γ can be obtained as follows:

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n+1)\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\int_{1}^{n+1}\frac{\mathrm{d}x}{x}\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\int_{k}^{k+1}\frac{\mathrm{d}x}{x}\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\int_{k}^{k+1}\frac{\mathrm{d}x}{x}\right).
\end{aligned} \tag{2}$$

Noting that

$$\frac{1}{k}=\int_{k}^{k+1}\frac{\mathrm{d}x}{\lfloor x\rfloor}, \tag{3}$$

where $\lfloor x \rfloor$ is the *floor function*, i.e. the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x, we can rewrite Eq. (2) as follows:

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\int_{k}^{k+1}\frac{\mathrm{d}x}{\lfloor x\rfloor}-\int_{k}^{k+1}\frac{\mathrm{d}x}{x}\right)\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{k}^{k+1}\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)\mathrm{d}x = \lim_{n\to\infty}\int_{1}^{n+1}\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)\mathrm{d}x. \qquad (4)\end{aligned}$$

Next, note that

$$\int_{1}^{n+1}\frac{\lceil x\rceil}{x^2}\,\mathrm{d}x = \int_{1}^{n+1}\frac{1}{\lfloor x\rfloor}\,\mathrm{d}x. \qquad (5)$$

Here $\lceil x \rceil$ is the *ceiling function*, i.e. the function that takes as input a real number x, and gives as output the smallest integer greater than or equal to x.

$$\lceil x\rceil = 1 + \lfloor x\rfloor. \qquad (6)$$

Indeed,

$$\begin{aligned}\int_{1}^{n+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{\lfloor x\rfloor}\right)\mathrm{d}x &= \sum_{k=1}^{n}\int_{k}^{k+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{\lfloor x\rfloor}\right)\mathrm{d}x\\ &= \sum_{k=1}^{n}\int_{k}^{k+1}\left(\frac{k+1}{x^2}-\frac{1}{k}\right)\mathrm{d}x = \sum_{k=1}^{\infty}\left[-\frac{k+1}{x}-\frac{x}{k}\right]_{x=k}^{x=k+1}\\ &= \sum_{k=1}^{n}\left[-\frac{k+1}{k+1}-\frac{k+1}{k}+\frac{k+1}{k}+\frac{k}{k}\right] = \sum_{k=1}^{n}0 = 0. \qquad (7)\end{aligned}$$

Thus, we can rewrite the integral for γ Eq. (4) as follows:

$$\gamma = \lim_{n\to\infty}\int_{1}^{n+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{x}\right)\mathrm{d}x = \int_{1}^{\infty}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{x}\right)\mathrm{d}x. \qquad (8)$$

Monte Carlo algorithm

A Monte Carlo algorithm to calculate the Euler-Mascheroni constant by using uniform random variables and elementary functions, is as follows:

1. Generate uniformly distributed on $(0,1]$ random numbers u_i, $i = 1,\dots,n$.
2. For every u_i calculate
$$w_i = 1 - \left\{\frac{1}{u_i}\right\}, \tag{9}$$
where $\{x\}$ denotes the fractional part of x,
$$\{x\} = x - \lfloor x \rfloor. \tag{10}$$
3. Calculate $\bar{w}(n)$ the average of w_i, $i = 1,\dots,n$.
$$\bar{w}(n) \equiv \left(\frac{1}{n}\sum_{i=1}^{n} w_i\right). \tag{11}$$
4. The Euler-Mascheroni constant, γ, is
$$\gamma = \lim_{n\to\infty} \bar{w}(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left\{\frac{1}{u_i}\right\}\right). \tag{12}$$

Indeed, for the unit uniform distribution,
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left\{\frac{1}{u_i}\right\}\right) = \int_0^1 \left(1 - \left\{\frac{1}{u}\right\}\right) \mathrm{d}u. \tag{13}$$

To evaluate the last integral, we introduce a new integration variable, x:
$$x = \frac{1}{u}, \quad \infty > x \geq 1, \quad u = \frac{1}{x}, \quad \mathrm{d}u = -\frac{\mathrm{d}x}{x^2}. \tag{14}$$

Then,
$$\begin{aligned}\lim_{n\to\infty} \bar{w}(n) &= -\int_\infty^1 (1 - \{x\})\frac{\mathrm{d}x}{x^2} = \int_1^\infty \frac{1 - (x - \lfloor x \rfloor)}{x^2}\mathrm{d}x \\ &= \int_1^\infty \left(\frac{1 + \lfloor x \rfloor}{x^2} - \frac{1}{x}\right)\mathrm{d}x = \int_1^\infty \left(\frac{\lceil x \rceil}{x^2} - \frac{1}{x}\right)\mathrm{d}x = \gamma.\end{aligned} \tag{15}$$

References

[1] Statistics Stack Exchange. *Estimate the Euler–Mascheroni constant by Monte Carlo simulations*. [Online; accessed 2023-11-24]. 2021. URL: https://stats.stackexchange.com/questions/531538/.